

Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ under gradient perturbation

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Abstract

For $\alpha \in (0, 2)$ and $M > 0$, we consider a family of nonlocal operators $\{\Delta + a^\alpha \Delta^{\alpha/2}, a \in (0, M]\}$ on \mathbb{R}^d under Kato class gradient perturbation. We establish the existence and uniqueness of their fundamental solutions, and derive their sharp two-sided estimates. The estimates give explicit dependence on a and recover the sharp estimates for Brownian motion with drift as $a \rightarrow 0$. Each fundamental solution determines a conservative Feller process X . We characterize X as the unique solution of the corresponding martingale problem as well as a Lévy process with singular drift.

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1 Introduction

Let B be a Brownian motion on \mathbb{R}^d with $\mathbb{E}[(B_t - B_0)^2] = 2t$, and Y be a rotationally symmetric α -stable process on \mathbb{R}^d that is independent of B . Here $d \geq 1$ and $\alpha \in (0, 2)$. Then $B + Y$ is a symmetric Lévy process that has both diffusive and jumping components. Let b be a bounded \mathbb{R}^d -valued function on \mathbb{R}^d . Using Girsanov transform, it is easy to show that for every $a > 0$, there is a strong Markov process $X^{a,b}$ on \mathbb{R}^d so that

$$dX_t^{a,b} = dZ_t^a + b(X_t^{a,b})dt, \quad (1.1)$$

where Z^a is a Lévy process that has the same distribution as $B + aY$. The goal of this paper is to study the transition density function $p^{a,b}(t, x, y)$ of the strong Markov process $X^{a,b}$ and its two-sided sharp estimates.

Recall that a rotationally symmetric α -stable process on \mathbb{R}^d is a Lévy process Y so that

$$\mathbb{E}_x[e^{i\xi(Y_t - Y_0)}] = e^{-t|\xi|^\alpha} \text{ for every } x, \xi \in \mathbb{R}^d \text{ and } t > 0.$$

The infinitesimal generator of Y is $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$, which is a prototype of nonlocal operator and can be written in the form

$$\Delta^{\alpha/2} f(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y| \geq \varepsilon} \mathcal{A}(d, -\alpha) \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy, \quad f \in C_c^2(\mathbb{R}^d). \quad (1.2)$$

Here $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma((d+\alpha)/2) \Gamma(1-\alpha/2)^{-1}$ is a normalizing constant, with $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$. Using Itô's formula, one can see that the infinitesimal generator of $X^{a,b}$ is

$$\mathcal{L}^{a,b} = \Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla.$$

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In this paper we will in fact study heat kernel estimates of $X^{a,b}$ not only for bounded drift function b but also for b in certain Kato class $\mathbb{K}_{d,1}$ which can be unbounded; see Definition 1.1. When b is in Kato class $\mathbb{K}_{d,1}$, one can not obtain the strong Markov process $X^{a,b}$ from $B + aY$ through Girsanov transform. So we will do it in the other way around. We first construct and establish in Theorem 1.2 the uniqueness of the fundamental solution $p^{a,b}(t, x, y)$ for operator $\mathcal{L}^{a,b}$, and obtain its two-sided sharp estimates in Theorem 1.3. The heat kernel $p^{a,b}(t, x, y)$ determines a conservative Feller process $X^{a,b}$. We then show in Theorem 1.5 that $X^{a,b}$ satisfies (1.1) through establishing the well-posedness of the martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ in Theorem 1.4. Moreover, we derive sharp two-sided estimates for $p^{a,b}(t, x, y)$ in such a way that gives the explicit dependence on a so that when $a \rightarrow 0$, we can recover the sharp two-sided heat kernel estimates for Brownian motion with drift obtained in Zhang [17, 18].

Brownian motions with drifts, which have $\Delta + b \cdot \nabla$ as their infinitesimal generators, have been studied by many authors under various conditions; see [13, 17, 18] and the references therein, where b belongs to some suitable Kato class. In [4], a fundamental solution to $\Delta^{\alpha/2} + b \cdot \nabla$ on \mathbb{R}^d with $d \geq 2$ is constructed and its two-sided estimates derived. The uniqueness of the fundamental solution, the well-posedness of the martingale problem for $(\Delta^{\alpha/2} + b \cdot \nabla, C_c^\infty(\mathbb{R}^d))$ and its connection to stochastic differential equations are recently settled in [11]. We also mention that relativistic stable processes with drifts have recently been studied in [12].

We now describe the main results of this paper in more details. The Lévy process Z^a has infinitesimal generator $\mathcal{L}^a := \Delta + a^\alpha \Delta^{\alpha/2}$, and Lévy intensity kernel

$$J^a(x, y) = a^\alpha \mathcal{A}(d, -\alpha) |x - y|^{-(d+\alpha)}, \quad (1.3)$$

The kernel $J^a(x, y)$ determines a Lévy system for Z^a , which describes the jumps of the process Z^a . Let $p^a(t, x, y) := p^a(t, x - y)$ be the transition density function of Z^a with respect to the Lebesgue measure on \mathbb{R}^d . Clearly, $p^a(t, z)$ is the smooth function determined by

$$\int_{\mathbb{R}^d} p^a(t, z) e^{iz \cdot \xi} dz = e^{-t(|\xi|^2 + a^\alpha |\xi|^\alpha)}, \quad \xi \in \mathbb{R}^d. \quad (1.4)$$

The following sharp two-sided estimates on $p^a(t, z)$, as stated in [5, Theorem 1.1], follows directly from [9, Theorem 1.4] (see also [16, Theorem 2.13]) by scaling. There exist constants $C_i \geq 1$, $i = 1, 2$, so that for all $a \in (0, \infty)$ and $(t, z) \in (0, \infty) \times \mathbb{R}^d$,

$$\begin{aligned} C_1^{-1} (t^{-d/2} \wedge (a^\alpha t)^{-d/\alpha}) \wedge \left(t^{-d/2} e^{-C_2 |z|^2/t} + (a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|z|^{d+\alpha}} \right) \\ \leq p^a(t, z) \leq C_1 (t^{-d/2} \wedge (a^\alpha t)^{-d/\alpha}) \wedge \left(t^{-d/2} e^{-|z|^2/(C_2 t)} + (a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|z|^{d+\alpha}} \right). \end{aligned} \quad (1.5)$$

We can view $\mathcal{L}^{a,b}$ as the perturbation of \mathcal{L}^a by $b \cdot \nabla$. So intuitively, the fundamental solution $p^{a,b}(t, x, y)$ of $\mathcal{L}^{a,b}$ should be related to the fundamental solution $p^a(t, x, y)$ by the following formula

$$p^{a,b}(t, x, y) = p^a(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p^{a,b}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds \quad (1.6)$$

for $t > 0$ and $x, y \in \mathbb{R}^d$. The above relation is a folklore and is called Duhamel's formula in literature. Just as in [4, 18], applying (1.6) recursively, it is reasonable to conjecture that $\sum_{k=0}^\infty p_k^{a,b}(t, x, y)$, if convergent, is a solution of (1.6), where $p_0^{a,b}(t, x, y) = p^a(t, x, y)$ and

$$p_k^{a,b}(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_{k-1}^{a,b}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds \text{ for } k \geq 1. \quad (1.7)$$

We now give the definition of Kato class $\mathbb{K}_{d,1}$. For a function $f = (f_1, \dots, f_k) : \mathbb{R}^d \rightarrow \mathbb{R}^k$ and $d \geq 2$, define

$$M_f(r) = \sup_{x \in \mathbb{R}^d} \int_{|x-y| < r} \frac{|f(y)|}{|x-y|^{d-1}} dy \quad \text{for } r > 0.$$

Definition 1.1. A function $f = (f_1, \dots, f_k) : \mathbb{R}^d \rightarrow \mathbb{R}^k$ is said to be in Kato class $\mathbb{K}_{d,1}$ if $\lim_{r \downarrow 0} M_f(r) = 0$ when $d \geq 2$, and bounded if $d = 1$.

It is easy to see that any bounded function is in Kato class $\mathbb{K}_{d,1}$ and, for $d \geq 2$, $L^p(\mathbb{R}^d) \subset \mathbb{K}_{d,1}$ for any $p > d$ by Hölder inequality. On the other hand, any function in $\mathbb{K}_{d,1}$ is locally integrable on \mathbb{R}^d .

For an integer $k \geq 1$, let $C_c^k(\mathbb{R}^d)$ denote the space of all continuous functions on \mathbb{R}^d with compact supports that have continuous derivatives up to and including k th-order, and set $C_c^\infty(\mathbb{R}^d) = \bigcap_{k=1}^\infty C_c^k(\mathbb{R}^d)$. Denote by $C_\infty(\mathbb{R}^d)$ the space of continuous functions on \mathbb{R}^d vanishing at the infinity, equipped with supremum norm. The following are the first two main results of this paper.

Theorem 1.2. Suppose that $M > 0$ and $b = (b_1, \dots, b_d) \in \mathbb{K}_{d,1}$. For every $a \in (0, M]$, there is a unique positive jointly continuous function $p^{a,b}(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ that satisfies (1.6) with $p^{a,b}(t, x, y) \leq c_1 p^a(t, x, y)$ both on $(0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d$ for some constants $c_1, t_0 > 0$, and that

$$p^{a,b}(t + s, x, y) = \int_{\mathbb{R}^d} p^{a,b}(t, x, z) p^{a,b}(s, z, y) dz \quad \text{for } t, s > 0, x, y \in \mathbb{R}^d. \quad (1.8)$$

Moreover, the following hold.

- (i) There is a constant $t_* = t_*(d, \alpha, M, b) > 0$, depending on b only via the rate at which $M_b(r)$ goes to zero, such that

$$p^{a,b}(t, x, y) = \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) \quad \text{on } (0, t_*] \times \mathbb{R}^d \times \mathbb{R}^d, \quad (1.9)$$

where $p_k^{a,b}(t, x, y)$ is defined by (1.7).

- (ii) $p^{a,b}(t, x, y)$ satisfies (1.6) on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.
- (iii) (Conservativeness) $\int_{\mathbb{R}^d} p^{a,b}(t, x, y) dy = 1$ for every $t > 0$ and $x \in \mathbb{R}^d$.
- (iv) for every $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in C_\infty(\mathbb{R}^d)$,

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^d} \frac{P_t^{a,b} f(x) - f(x)}{t} g(x) dx = \int_{\mathbb{R}^d} \mathcal{L}^{a,b} f(x) g(x) dx, \quad (1.10)$$

where $P_t^{a,b} f(x) = \int_{\mathbb{R}^d} p^{a,b}(t, x, y) f(y) dy$.

Here and after, the meaning of the phrase “depending on b only via the rate at which $M_b(r)$ goes to zero” is that the statement is true for any \mathbb{R}^d -valued function \tilde{b} on \mathbb{R}^d with $M_{\tilde{b}}(r) \leq M_b(r)$ for all $r > 0$. In this paper, we use $:=$ as a way of definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For constants $a, \beta > 0$, we define

$$q_{d,\beta}^a(t, z) = t^{-d/2} \exp\left(-\frac{\beta|z|^2}{t}\right) + t^{-d/2} \wedge \frac{a^\alpha t}{|z|^{d+\alpha}} \quad \text{for } t > 0, z \in \mathbb{R}^d. \quad (1.11)$$

Theorem 1.3. For every $M > 0$ and $T > 0$, there are constants $C_i = C_i(d, \alpha, M)$, $i = 4, 6$ and $C_j = C_j(d, \alpha, M, T, b)$, $j = 3, 5$ depending on b only via the rate at which $M_b(r)$ goes to zero, such that for all $a \in (0, M]$ and $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_3 q_{d,C_4}^a(t, x - y) \leq p^{a,b}(t, x, y) \leq C_5 q_{d,C_6}^a(t, x - y). \quad (1.12)$$

The heat kernel upper bound estimate of $p^{a,b}(t, x, y)$ is obtained by estimating each $p_k^{a,b}(t, x, y)$ in (1.9). It relies on a key estimate obtained in Lemma 3.2, which can be regarded as an analogy of the so called 3P estimate in [18, Lemma 3.1] and [4, Lemma 13]. However, unlike the case in [18] where there is only Gaussian term coming from Brownian motion and the case in [4] where there is only polynomial term coming from symmetric stable process, there are many new difficulties to overcome as we have to deal with a mixture of them. It seems to be difficult to establish the positivity of $p^{a,b}(t, x, y)$ directly from the estimates of $p_k^{a,b}(t, x, y)$ as was done in [4] for the symmetric stable process case. Following [10], we derive the positivity of $p^{a,b}(t, x, y)$ by using the Hille-Yosida-Ray theorem when b is bounded and continuous. For general b in Kato class $\mathbb{K}_{d,1}$, we approximate b by a sequence of smooth b_n . For the lower bound of $p^{a,b}(t, x, y)$ in Theorem 1.3, we identify and use the Lévy system of the Feller process $\{X_t^{a,b}, t \geq 0, \mathbb{P}_x^{a,b}, x \in \mathbb{R}^d\}$ associated with $\{P_t^{a,b}, t \geq 0\}$ to get the polynomial part (see Lemma 5.6), and use a chaining argument to get the Gaussian part (see Lemma 5.7).

Let $\mathbb{D}([0, \infty), \mathbb{R}^d)$ be the space of right continuous \mathbb{R}^d -valued functions on $[0, \infty)$ having left limits equipped with Skorokhod topology, and let X_t be the coordinate map on $\mathbb{D}([0, \infty), \mathbb{R}^d)$. A probability measure \mathbf{Q} on $\mathbb{D}([0, \infty), \mathbb{R}^d)$ is said to be a solution to the martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ with initial value $x \in \mathbb{R}^d$ if $\mathbf{Q}(X_0 = x) = 1$ and for every $f \in C_c^\infty(\mathbb{R}^d)$ and $t > 0$, $\int_0^t |\mathcal{L}^{a,b} f(X_s)| ds < \infty$ \mathbf{Q} -a.s. and

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}^{a,b} f(X_s) ds$$

is a \mathbf{Q} -martingale. The martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ with initial value $x \in \mathbb{R}^d$ is said to be well-posed if it has a unique solution.

Theorem 1.4. *The martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ is well-posed for every initial value $x \in \mathbb{R}^d$. These martingale problem solutions $\{\mathbb{P}_x, x \in \mathbb{R}^d\}$ form a strong Markov process X , which has $p^{a,b}(t, x, y)$ of Theorem 1.2 as its transition density function with respect to the Lebesgue measure on \mathbb{R}^d .*

We now connect the strong Markov process in Theorem 1.4 to solution of SDE (1.1).

Theorem 1.5. *For each $x \in \mathbb{R}^d$, SDE (1.1) has a unique weak solution with initial value x . Moreover, weak solutions with different starting points can be constructed on $\mathbb{D}([0, \infty), \mathbb{R}^d)$, and the process Z^a in (1.1) can be chosen in such a way that it is the same for all starting point $x \in \mathbb{R}^d$. The law of the weak solution to (1.1) is the unique solution to the martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$.*

Remark 1.6. Brownian motion with measure-valued singular drift on \mathbb{R}^d , where $d \geq 2$ and the \mathbb{R}^d -valued drift b is replaced by a measure $\mu = (\mu_1, \dots, \mu_d)$ in Kato class $\mathbb{K}_{d,1}$, is introduced and constructed in Bass and Chen [2]. With the two-sided heat kernel estimates from Theorem 1.3, one can easily construct Lévy process Z^a with singular measure-valued drift $\mu = (\mu_1, \dots, \mu_d)$ in the sense of [2] and obtain its sharp two-sided heat kernel estimates. The key is to note that the two-sided heat kernel estimates in Theorem 1.3 depend on the drift b only through its upper bound of $M_b(r)$ so we can approximate the measure-valued drift μ by a sequence of function-valued drifts whose Kato norms are uniformly controlled by that of μ . Here are the details. Suppose $\mu = (\mu_1, \dots, \mu_d) \in K_{d,1}$, that is,

$$\lim_{r \rightarrow 0} M_\mu(r) := \lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|y-x| < r} \frac{1}{|y-x|^{d-1}} |\mu|(dy) = 0.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. We can approximate μ by $b_n(x) dx = \varphi_n * \mu(dx)$, where $\varphi_n(x) = n^d \varphi(nx)$. Note that $\{b_n; n \geq 1\} \subset \mathbb{K}_{d,1}$ with $M_{b_n}(r) \leq M_\mu(r)$ for every $n \geq 1$ and $r > 0$. Denote by $p^{b_n}(t, x, y)$ and X^n the heat kernel for \mathcal{L}^{b_n} and its

corresponding Feller process. The two-sided heat kernel estimate (1.12) holds uniformly in n for $p^{a,b_n}(t, x, y)$ on $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$. Similar to that of [15, Theorem 3.9], one can show that $\{p^{a,b_n}(t, x, y); t > 0, x, y \in \mathbb{R}^d\}$ converges locally uniformly to $p^{a,\mu}(t, x, y)$. It is easy to verify that $p^{a,\mu}(t, x, y)$ is a positive kernel which enjoys the two-sided estimates (1.12). Moreover, it satisfies the Chapman-Kolmogorov equation and $\int_{\mathbb{R}^d} p^{a,\mu}(t, x, y) dy = 1$ for all $t > 0$ and $x \in \mathbb{R}^d$. The kernel $p^{a,\mu}(t, x, y)$ determines a Feller process X . It is not hard to verify that it is a Lévy process Z^a with measure-valued drift μ in the sense of Bass and Chen [2]. See [15] for the case when Z^a is a rotationally symmetric stable process on \mathbb{R}^d . \square

The rest of this paper is organized as follows. In Section 2, we recall some properties of $p^a(t, x, y)$ and derive its gradient estimates, as well as properties of functions in Kato class $\mathbb{K}_{d,1}$. In Section 3, we construct $p^{a,b}(t, x, y)$ using the series of $p_k^{a,b}(t, x, y)$ and prove Theorem 1.2 through a series of lemmas except the positivity of $p^{a,b}(t, x, y)$. In addition, we derive the upper bound of $|p^{a,b}(t, x, y)|$. The positivity of $p^{a,b}(t, x, y)$ is shown in Section 4, where we use the fact that $\{P_t^{a,b}, t \geq 0\}$ is Feller semigroup, that is, a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$. In Section 5, we determine the Lévy system of the Feller process $X^{a,b}$ associated with the Feller semigroup $\{P_t^{a,b}, t \geq 0\}$. We then use it to derive the lower bound estimate of $p^{a,b}(t, x, y)$. In Section 6, we prove Theorem 1.4 and Theorem 1.5.

For convenience, in the rest of this paper, we assume $d \geq 2$. When $d = 1$, it can be treated in a similar but simpler way as the drift b would be bounded. Throughout this paper, unless stated otherwise, we use C_1, C_2, \dots , to denote positive constants whose value are fixed throughout the paper, while using c_1, c_2, \dots , to denote positive constants whose exact value are unimportant and whose value can change from one appearance to another. We use notation $c = c(d, \alpha, \dots)$ to indicate that this constant depends only on d, α, \dots . For two non-negative functions f, g , the notation $f \stackrel{c}{\lesssim} g$ means that $f \leq cg$ on their common domains of definition while $f \stackrel{c}{\asymp} g$ means that $c^{-1}g \leq f \leq cg$. We also write mere \lesssim and \asymp if c is unimportant or understood. For reader's convenience, we summarize the notation of functions that will appear many times throughout this paper. For $t > 0$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} p^a(t, x, y) &= p^a(t, x - y) : \text{the transition density function of } B + aY \\ g_{d,\beta}(t, x, y) &= g_{d,\beta}(t, x - y) := t^{-d/2} \exp\left(-\frac{\beta|x - y|^2}{t}\right), \end{aligned} \quad (1.13)$$

$$\begin{aligned} g_d(t, x, y) &= g_d(t, x - y) := (4\pi)^{-d/2} g_{d,1/4}(t, x - y), \\ q_{d,\beta}^a(t, x, y) &= q_{d,\beta}^a(t, x - y) := g_{d,\beta}(t, x - y) + t^{-d/2} \wedge \frac{a^\alpha t}{|x - y|^{d+\alpha}}. \end{aligned} \quad (1.14)$$

2 Preliminaries

The following is a direct consequence of (1.5); see [5, Corollary 1.2].

Theorem 2.1. *For any $M > 0$ and $T > 0$, there exist constants $C_i, i = 8, 10$ and $C_j = C_j(d, \alpha, M, T)$, $j = 7, 9$ such that for all $a \in (0, M]$ and $(t, x) \in (0, T] \times \mathbb{R}^d$,*

$$C_7 q_{d,C_8}^a(t, x) \leq p^a(t, x) \leq C_9 q_{d,C_{10}}^a(t, x).$$

It is easy to see that for any $\theta > 0$, there is a positive constant $c_1 = c_1(d, \beta, \theta)$ such that

$$g_{d,\beta}(t, x) \leq t^{-d/2} \wedge \frac{c_1 t^\theta}{|x|^{d+2\theta}}, \quad t > 0 \text{ and } x \in \mathbb{R}^d, \quad (2.1)$$

which will be frequently used in the rest of this paper.

Recall the definition of $q_{d,\beta}^a(t, x)$ in (1.11). There is a constant $C_{11} = C_{11}(\alpha, M, T, \beta)$ such that for all $a \in (0, M]$ and all $(t, z) \in (0, T] \times \mathbb{R}^d$,

$$q_{d,\beta}^a(t, z) \stackrel{C_{11}}{\gtrsim} g_{d,\beta}(t, z) + \frac{a^\alpha t}{|z|^{d+\alpha}} \mathbf{1}_{\{|z|^2 \geq t\}}. \quad (2.2)$$

Indeed, $t^{-d/2} \wedge \frac{a^\alpha t}{|z|^{d+\alpha}} \leq t^{-d/2} \leq e^\beta g_{d,\beta}(t, z)$ when $|z|^2 < t$. Thus

$$q_{d,\beta}^a(t, z) \stackrel{e^\beta+1}{\lesssim} g_{d,\beta}(t, z) + \frac{a^\alpha t}{|z|^{d+\alpha}} \mathbf{1}_{\{|z|^2 \geq t\}} \quad \text{for } a, t > 0 \text{ and } z \in \mathbb{R}^d. \quad (2.3)$$

On the other hand, for $a \in (0, M]$ and $t \in (0, T]$,

$$\frac{a^\alpha t}{|z|^{d+\alpha}} \leq M^\alpha t^{-d/2+1-\alpha/2} \leq M^\alpha T^{1-\alpha/2} t^{-d/2} \quad \text{if } |z|^2 \geq t,$$

and so

$$g_{d,\beta}(t, z) + \frac{a^\alpha t}{|z|^{d+\alpha}} \mathbf{1}_{\{|z|^2 \geq t\}} \stackrel{M^\alpha T^{1-\alpha/2} \vee 1}{\lesssim} q_{d,\beta}^a(t, z). \quad (2.4)$$

The claim (2.2) now follows from (2.3) and (2.4) with $C_{11} = (e^\beta + 1) \vee (M^\alpha T^{1-\alpha/2} \vee 1)$.

When there is no danger of confusion, for $x \in \mathbb{R}^d$ and integer $k \geq 1$, for simplicity, we write $q_{d+k,\beta}^a(t, x)$ for $q_{d+k,\beta}^a(t, \tilde{x})$, where $\tilde{x} := (x, 0, \dots, 0) \in \mathbb{R}^{d+k}$. Same convention will apply to function $g_{d,\beta}(t, x)$.

The following theorem gives the two-sided estimate of $|\nabla_x p^a(t, x)|$. In this paper, only its upper bound will be used.

Theorem 2.2. *For any $M > 0$ and $T > 0$, there is a positive constant $C_{12} = C_{12}(d, \alpha, M, T)$ such that for all $a \in (0, M]$ and $(t, x) \in (0, T] \times \mathbb{R}^d$,*

$$2\pi C_7 q_{d+2, C_8}^a(t, x) |x| \leq |\nabla_x p^a(t, x)| \leq C_{12} q_{d+1, 3C_{10}/4}^a(t, x).$$

Proof. It is well-known that, for each $t > 0$, $x \mapsto p^a(t, x)$ attains its maximum at $x = 0$ so we have $\nabla p^a(t, 0) = 0$. So it suffices to consider $x \in \mathbb{R}^d \setminus \{0\}$. Recall that $g_d(t, z) = (4\pi t)^{-d/2} e^{-|z|^2/(4t)}$, which is the transition density function of Brownian motion B . Let S_t be the $\alpha/2$ -stable subordinator at time t , independent of B , and $\eta_t^a(u)$ be the density function of $a^2 S_t$. The Lévy process Z^a can be realized as a subordination of Brownian motion B ; that is, $\{Z_t^a; t \geq 0\}$ has the same distribution as $\{B_{t+a^2 S_t}; t \geq 0\}$. Thus

$$p^a(t, x) = \int_t^{+\infty} g_d(u, x) \mathbb{P}(t + a^2 S_t \in du) = \int_t^{+\infty} g_d(u, x) \eta_t^a(u - t) du,$$

and so

$$\nabla_x p^a(t, x) = \nabla_x \int_t^\infty g_d(u, x) \eta_t^a(u - t) du.$$

Let $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is on j^{th} place. Let $x \in \mathbb{R}^d \setminus \{0\}$ and set $s \in (-|x|/2, |x|/2)$. By the mean-value theorem, there exists $\xi \in (-|s|, |s|)$ such that

$$\begin{aligned} \left| \frac{g_d(u, x + s e_j) - g_d(u, x)}{s} \right| &= \left| \frac{\partial}{\partial x_j} g_d(u, x + \xi e_j) \right| = \left| \frac{x_j + \xi}{2u} g_d(u, x + \xi e_j) \right| \\ &\leq \frac{|x| g_d(u, x/2)}{u} \leq c(d) |x|^{-d-1}, \end{aligned}$$

where $c(d)$ is a positive constant depending only on d . Since $\int_t^\infty c(d)|x|^{-d-1}\eta_t(u-t)du < \infty$, we have by the dominated convergence theorem

$$\nabla_x p^a(t, x) = \int_t^\infty \nabla_x g_d(u, x) \eta_t^a(u-t) du = \int_t^\infty -\frac{x g_d(u, x)}{2u} \eta_t^a(u-t) du = -2\pi x p_{(d+2)}^a(t, \tilde{x}), \quad (2.5)$$

where $\tilde{x} := (x, 0, 0) \in \mathbb{R}^{d+2}$ and $p_{d+2}^a(t, \tilde{x})$ is the transition density function of Z^a in dimension $d+2$. Thus, by Theorem 2.1, we have

$$2\pi C_7 q_{d+2, C_8}^a(t, x)|x| \leq |\nabla_x p^a(t, x)| \leq 2\pi C_9 q_{d+2, C_{10}}^a(t, x)|x|. \quad (2.6)$$

Note that for all $t > 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} t^{-(d+2)/2} \exp\left(-\frac{C_{10}|x|^2}{t}\right) |x| &= t^{-(d+1)/2} \exp\left(-\frac{3C_{10}}{4} \frac{|x|^2}{t}\right) \cdot \frac{|x|}{t^{1/2}} \exp\left(-\frac{C_{10}}{4} \frac{|x|^2}{t}\right) \\ &\leq \sqrt{\frac{2}{C_{10}e}} t^{-(d+1)/2} \exp\left(-\frac{3C_{10}}{4} \frac{|x|^2}{t}\right). \end{aligned}$$

This together with (2.2) and (2.6) proves the theorem with $C_{12} := 2\pi C_9 C_{11} \left(\sqrt{2/(C_{10}e)} \vee 1\right)$. \square

For $\beta > \frac{1}{2}$ and a function f on \mathbb{R}^d , define for $r > 0$ and $x \in \mathbb{R}^d$,

$$H^\beta(r, x) = \frac{1}{|x|^{d-1}} \wedge \frac{r^\beta}{|x|^{d-1+2\beta}} \quad \text{and} \quad H_f^\beta(r, x) = \int_{\mathbb{R}^d} |f(y)| H^\beta(r, x-y) dy.$$

Lemma 2.3. Assume $\beta > \frac{1}{2}$. There is a constant $C_{13} = C_{13}(d, \beta)$ so that

$$M_f(\sqrt{r}) \leq H_f^\beta(r, x) \leq C_{13} M_f(\sqrt{r}), \quad (2.7)$$

for every $r > 0, x \in \mathbb{R}^d$ and for every f on \mathbb{R}^d . Consequently, $f \in \mathbb{K}_{d,1}$ if and only if

$$\limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} H_f^\beta(r, x) = 0.$$

The lower bound in (2.7) is trivial. The proof of the upper bound in (2.7) is almost the same as that for [4, Lemma 11 and Corollary 12] except with 2 in place of α there. So we omit its details.

Let

$$N^\beta(r, x) = \int_0^r g_{d+1, \beta}(s, x) ds = \int_0^r s^{-(d+1)/2} \exp\left(-\frac{\beta|x|^2}{s}\right) ds, \quad r > 0, x \in \mathbb{R}^d.$$

Lemma 2.4. $f \in \mathbb{K}_{d,1}$ if and only if

$$\limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)| N^\beta(r, x-y) dy = 0 \quad \text{for all } \beta > 0. \quad (2.8)$$

Proof. Condition (2.8) is introduced in [17]. Its equivalence to the $\mathbb{K}_{d,1}$ condition is proved in [14, Proposition 2.3]. For reader's convenience, we give a short proof here.

By a change of variable $t = \beta|x|^2/s$, we have

$$N^\beta(r, x) = \frac{1}{\beta^{(d-1)/2} |x|^{d-1}} \int_{\beta|x|^2/r}^\infty t^{(d-3)/2} e^{-t} dt. \quad (2.9)$$

Thus

$$c_1(d, \beta) \frac{1}{|x|^{d-1}} \mathbf{1}_{\{|x| \leq \sqrt{r}\}} \leq N^\beta(r, x) \leq c_2(d, \beta) H^1(r, x) \quad (2.10)$$

The equivalence now follows from Lemma 2.3. \square

3 Construction and upper bound estimates

By [18, Lemma 3.1] and its proof, we have the following lemma. Recall that $g_{d,\beta}(t, x, y) := g_{d,\beta}(t, x - y)$ is defined by (1.13), and define $H^\beta(r, x, y) = H^\beta(r, x - y)$ and $N^\beta(r, x, y) = N^\beta(r, x - y)$.

Lemma 3.1. *For any $0 < \beta_1 < \beta_2 < \infty$, there exist constants $C_g = C_g(d, \beta_1/\beta_2)$ and $C_\beta = \min\{\beta_2 - \beta_1, \beta_1/2\}$ such that for all $t > 0$ and $x, y, z \in \mathbb{R}^d$,*

$$\int_0^t g_{d,\beta_1}(t-s, x, z) s^{-1/2} g_{d,\beta_2}(s, z, y) ds \leq C_g (N^{C_\beta}(t, x, z) + N^{C_\beta}(t, z, y)) g_{d,\beta_1}(t, x, y).$$

In the rest of this paper, we assume $b \in \mathbb{K}_{d,1}$ and let $\gamma = (1 + \alpha \wedge 1)/2$. The following lemma plays an important role in this paper and it is an analogy of [4, Lemma 13] or [18, Lemma 3.1].

Lemma 3.2. *Suppose $M > 0$ and $T > 0$. For any $0 < \beta_1 < \beta_2 < \infty$, there is a positive constant $C_{14} = C_{14}(d, \alpha, M, T, \beta_1, \beta_2)$ such that for all $a \in (0, M]$ and $(t, x, y, z) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\int_0^t q_{d,\beta_1}^a(t-s, x, z) q_{d+1,\beta_2}^a(s, z, y) ds \leq C_{14} (H^\gamma(t, x, z) + H^\gamma(t, z, y)) q_{d,\beta_1}^a(t, x, y). \quad (3.1)$$

Consequently, there is a positive constant $C_{15} = C_{15}(d, \alpha, M, T)$ such that for all $a \in (0, M]$ and $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\int_0^t \int_{\mathbb{R}^d} q_{d,\beta_1}^a(t-s, x, z) |b(z)| q_{d+1,\beta_2}^a(s, z, y) dz ds \leq C_{15} M_b(\sqrt{t}) q_{d,\beta_1}^a(t, x, y), \quad (3.2)$$

Proof. We first verify (3.1). By (2.2), for all $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, there is a constant $c_1 = c_1(\alpha, M, T, \beta_1, \beta_2)$ such that

$$\begin{aligned} I &:= \int_0^t q_{d,\beta_1}^a(t-s, x, z) q_{d+1,\beta_2}^a(s, z, y) ds \\ &\stackrel{c_1}{\lesssim} \int_0^t \left(g_{d,\beta_1}(t-s, x, z) + \frac{a^\alpha(t-s)}{|x-z|^{d+\alpha}} \mathbf{1}_{\{|x-z|^2 \geq t-s\}} \right) \left(\frac{g_{d,\beta_2}(s, z, y)}{s^{1/2}} + \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} \mathbf{1}_{\{|z-y|^2 \geq s\}} \right) ds \\ &= \int_0^t g_{d,\beta_1}(t-s, x, z) \frac{g_{d,\beta_2}(s, z, y)}{s^{1/2}} ds + \int_0^t g_{d,\beta_1}(t-s, x, z) \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} \mathbf{1}_{\{|z-y|^2 \geq s\}} ds \\ &\quad + \int_0^t \frac{a^\alpha(t-s)}{|x-z|^{d+\alpha}} \mathbf{1}_{\{|x-z|^2 \geq t-s\}} \frac{g_{d,\beta_2}(s, z, y)}{s^{1/2}} ds \\ &\quad + \int_0^t \frac{a^\alpha(t-s)}{|x-z|^{d+\alpha}} \mathbf{1}_{\{|x-z|^2 \geq t-s\}} \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} \mathbf{1}_{\{|z-y|^2 \geq s\}} ds \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We will treat each term separately. First, by Lemma 3.1, there are constants $c_2 = c_2(d, \beta_1/\beta_2)$ and $c_3 = c_3(\beta_2 - \beta_1, \beta_1/2)$ such that $I_1 \leq c_2 (N^{c_3}(t, x, z) + N^{c_3}(t, z, y)) g_{d,\beta_1}(t, x, y)$, while

$$\begin{aligned} I_2 &= \left(\int_0^{t/2} + \int_{t/2}^t \right) g_{d,\beta_1}(t-s, x, z) \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} \mathbf{1}_{\{|z-y|^2 \geq s\}} ds \\ &\stackrel{2^{d+\alpha} M^\alpha}{\lesssim} t^{-d/2} \int_0^{(t/2) \wedge |z-y|^2} \frac{s}{|z-y|^{d+1+\alpha}} ds \\ &\quad + t^{-d/2} \mathbf{1}_{\{|z-y|^2 \geq t/2\}} \int_0^{t/2} s^{-d/2} \exp\left(-\frac{\beta_1|x-z|^2}{s}\right) \frac{t^{1-\alpha/2}}{|z-y|} ds \\ &\stackrel{2}{\lesssim} t^{-d/2} \left(\frac{1}{|z-y|^{d-3+\alpha}} \wedge \frac{t^2}{|z-y|^{d+1+\alpha}} \right) \end{aligned}$$

$$\begin{aligned}
& + t^{-d/2+1-\alpha/2} \int_0^{t/2} s^{-(d+1)/2} \exp\left(-\frac{\beta_1|x-z|^2}{s}\right) ds \\
& \stackrel{T^{1-\alpha/2}}{\lesssim} t^{-d/2} \left(\frac{1}{|z-y|^{d-1}} \wedge \frac{t^{(1+\alpha)/2}}{|z-y|^{d+\alpha}} \right) + t^{-d/2} N^{\beta_1}(t, x, z) \\
& = t^{-d/2} \left(N^{\beta_1}(t, x, z) + H^{(1+\alpha)/2}(t, z, y) \right). \tag{3.3}
\end{aligned}$$

On the other hand, if $|x-z| \geq |z-y|$, then $2|x-z| \geq |x-z| + |z-y| \geq |x-y|$, and so

$$\begin{aligned}
I_2 & \stackrel{c_4(d,\alpha,\beta_1)}{\lesssim} \int_0^{t \wedge |z-y|^2} \frac{(t-s)^{\alpha/2}}{|x-z|^{d+\alpha}} \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} ds \\
& \stackrel{c_5(d,\alpha)}{\lesssim} \frac{a^\alpha t}{|x-y|^{d+\alpha}} \int_0^{t \wedge |z-y|^2} t^{\alpha/2-1} \frac{s}{|z-y|^{d+1+\alpha}} ds \\
& \stackrel{2}{\lesssim} \frac{a^\alpha t}{|x-y|^{d+\alpha}} t^{\alpha/2-1} \frac{t^2 \wedge |z-y|^4}{|z-y|^{d+1+\alpha}} \\
& \leq \frac{a^\alpha t}{|x-y|^{d+\alpha}} H^{1+\alpha/2}(t, z, y). \tag{3.4}
\end{aligned}$$

If $|x-z| < |z-y|$, then $2|z-y| \geq |x-y|$ and

$$\begin{aligned}
I_2 & \stackrel{2^{d+\alpha}}{\lesssim} \frac{a^\alpha}{|x-y|^{d+\alpha}} \int_0^t g_{d,\beta_1}(t-s, x, z) \sqrt{s} ds \\
& = \frac{a^\alpha t}{|x-y|^{d+\alpha}} \int_0^t \frac{\sqrt{s} \sqrt{t-s}}{t} (t-s)^{-(d+1)/2} \exp\left(-\frac{\beta_1|x-z|^2}{t-s}\right) ds \\
& \leq \frac{a^\alpha t}{|x-y|^{d+\alpha}} N^{\beta_1}(t, x, z). \tag{3.5}
\end{aligned}$$

Thus we have by (3.3)-(3.5)

$$I_2 \stackrel{c_6(d,\alpha,\beta_1,M,T)}{\lesssim} \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}} \right) \left(N^{\beta_1}(t, x, z) + H^{(1+\alpha)/2}(t, z, y) \right).$$

Similarly, we have

$$I_3 \stackrel{c_7(d,\alpha,\beta_2,M,T)}{\lesssim} \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}} \right) \left(H^{1+\alpha/2}(t, x, z) + N^{\beta_2}(t, z, y) \right).$$

It remains to estimate I_4 . If $|x-z|^2 \geq t-s$ and $|z-y|^2 \geq s$, then $|x-z| \vee |z-y| \geq \sqrt{t/2}$. Since $|x-z| \vee |z-y| \geq |x-y|/2$, we have $|x-z| \vee |z-y| \geq \frac{1}{2}(\sqrt{t} \vee |x-y|)$. Therefore

$$\begin{aligned}
& \frac{t-s}{|x-z|^{d+\alpha}} \frac{s}{|z-y|^{d+1+\alpha}} \\
& = \frac{1}{(|x-z| \vee |z-y|)^{d+\alpha}} \frac{1}{(|x-z| \wedge |z-y|)^{d+\alpha}} \frac{(t-s)s}{|z-y|} \\
& \stackrel{2^{d+\alpha}}{\lesssim} \frac{1}{(\sqrt{t} \vee |x-y|)^{d+\alpha}} \frac{(t-s)s}{(|x-z| \wedge |z-y|)^{d+1+\alpha}} \\
& \leq \left(t^{-d/2+1-\alpha/2} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \frac{(t-s)s}{t} \left(\frac{1}{|x-z|^{d+1+\alpha}} + \frac{1}{|z-y|^{d+1+\alpha}} \right) \\
& \stackrel{(T \vee 1)^{1-\alpha/2}}{\lesssim} \left(t^{-d/2} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \left(\frac{t-s}{|x-z|^{d+1+\alpha}} + \frac{s}{|z-y|^{d+1+\alpha}} \right)
\end{aligned}$$

Thus,

$$\begin{aligned}
I_4 &= a^{2\alpha} \int_0^t \mathbf{1}_{\{|x-z|^2 \geq t-s\}} \mathbf{1}_{\{|z-y|^2 \geq s\}} \frac{t-s}{|x-z|^{d+\alpha}} \frac{s}{|z-y|^{d+1+\alpha}} ds \\
&\stackrel{c_8(d,\alpha,T)}{\lesssim} a^{2\alpha} \left(t^{-d/2} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \int_0^t \mathbf{1}_{\{|x-z|^2 \geq t-s\}} \mathbf{1}_{\{|z-y|^2 \geq s\}} \\
&\quad \times \left(\frac{t-s}{|x-z|^{d+1+\alpha}} + \frac{s}{|z-y|^{d+1+\alpha}} \right) ds \\
&\stackrel{M^\alpha(M \vee 1)^\alpha}{\lesssim} \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}} \right) \int_0^t \mathbf{1}_{\{|x-z|^2 \geq t-s\}} \mathbf{1}_{\{|z-y|^2 \geq s\}} \\
&\quad \times \left(\frac{t-s}{|x-z|^{d+1+\alpha}} + \frac{s}{|z-y|^{d+1+\alpha}} \right) ds.
\end{aligned} \tag{3.6}$$

Notice that

$$\begin{aligned}
\int_0^t \frac{t-s}{|x-z|^{d+1+\alpha}} \mathbf{1}_{\{|x-z|^2 \geq t-s\}} ds &= \frac{1}{|x-z|^{d+1+\alpha}} \int_0^{t \wedge |x-z|^2} r dr \\
&\leq 2^{-1} t^{1-\alpha/2} \left(\frac{1}{|x-z|^{d-1}} \wedge \frac{t^{1+\alpha/2}}{|x-z|^{d+1+\alpha}} \right) \\
&\leq 2^{-1} T^{1-\alpha/2} H^{1+\alpha/2}(t, x, z).
\end{aligned} \tag{3.7}$$

Similarly,

$$\int_0^t \frac{s}{|z-y|^{d+1+\alpha}} \mathbf{1}_{\{|z-y|^2 \geq s\}} ds \leq 2^{-1} T^{1-\alpha/2} H^{1+\alpha/2}(t, z, y). \tag{3.8}$$

We have by (3.6), (3.7) and (3.8),

$$I_4 \leq c_9(d, \alpha, M, T) \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}} \right) \left(H^{1+\alpha/2}(t, x, z) + H^{1+\alpha/2}(t, z, y) \right).$$

Hence by (2.10) and the fact that $\beta \mapsto H^\beta(t, x, y)$ is decreasing, we have

$$I \stackrel{C_{14}(d,\alpha,\beta_1,\beta_2,M,T)}{\lesssim} (H^\gamma(t, x, z) + H^\gamma(t, z, y)) \left(g_{d,\beta_1}(t, x, y) + t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}} \right).$$

This completes the proof of (3.1). Multiplying the both sides of (3.1) by $|b(z)|$, we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} q_{d,\beta_1}^a(t-s, x, z) |b(z)| q_{d+1,\beta_2}^a(s, z, y) dz ds \\
&\stackrel{C_{14}}{\lesssim} \int_{\mathbb{R}^d} q_{d,\beta_1}^a(t, x, y) |b(z)| (H^\gamma(t, x, z) + H^\gamma(t, z, y)) dz \\
&\stackrel{2}{\lesssim} q_{d,\beta_1}^a(t, x, y) \sup_{x \in \mathbb{R}^d} H_b^\gamma(t, x) \stackrel{C_{13}}{\lesssim} M_b(\sqrt{t}) q_{d,\beta_1}^a(t, x, y).
\end{aligned}$$

This proves the lemma with $C_{15} = 2C_{14}C_{13}$. □

For $t > 0$ and $x, y \in \mathbb{R}^d$, we define

$$\begin{aligned}
|p|_0^{a,b}(t, x, y) &= p^a(t, x, y), \\
|p|_k^{a,b}(t, x, y) &= \int_0^t \int_{\mathbb{R}^d} |p|_{k-1}^{a,b}(t-s, x, z) |b(z)| |\nabla_z p^a(s, z, y)| dz ds, \quad \text{for } k \geq 1.
\end{aligned}$$

For every $M > 0$ and $T > 0$, we can verify by induction that

$$|p|_k^{a,b}(t, x, y) \leq C_9(C_{12}C_{15}M_b(\sqrt{t}))^k q_{d,C_{10}/2}^a(t, x, y), \quad a \in (0, M], (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (3.9)$$

Indeed, (3.9) holds for $k = 0$. Assume (3.9) holds for k . Then by assumption and (3.2),

$$\begin{aligned} |p|_{k+1}^{a,b}(t, x, y) &\leq C_9(C_{12}C_{15}M_b(\sqrt{t}))^k C_{12} \int_0^t \int_{\mathbb{R}^d} q_{d,C_{10}/2}^a(t-s, x, z) |b(z)| q_{d+1,3C_{10}/4}^a(s, z, y) dz ds \\ &\leq C_9(C_{12}C_{15}M_b(\sqrt{t}))^k C_{12}C_{15}M_b(\sqrt{t}) q_{d,C_{10}/2}^a(t, x, y) \\ &\leq C_9(C_{12}C_{15}M_b(\sqrt{t}))^{k+1} q_{d,C_{10}/2}^a(t, x, y). \end{aligned}$$

Thus for every $k \geq 1$, $t \in (0, T]$, $p_k^{a,b}(t, x, y)$ of (1.7) is well defined and has bound

$$|p_k^{a,b}(t, x, y)| \leq |p|_k^{a,b}(t, x, y) \leq C_9(C_{12}C_{15}M_b(\sqrt{t}))^k q_{d,C_{10}/2}^a(t, x, y) < \infty. \quad (3.10)$$

Lemma 3.3. *Suppose $M > 0$. For every $a \in (0, M]$ and $k \geq 0$, $p_k^{a,b}(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.*

Proof. We will use induction in k to prove this lemma. Obviously, $p_0^{a,b}(t, x, y) = p^a(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Assume $p_k^{a,b}(t, x, y)$ is jointly continuous. By (2.1) with $\theta = 1$,

$$q_{d,C_{10}/2}^a(t, x, y) \leq t^{-d/2} \wedge \frac{c_1 t}{|x-y|^{d+2}} + t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}}, \quad t > 0, x, y \in \mathbb{R}^d, \quad (3.11)$$

for some positive constant c_1 depending only on d . Suppose $T > 1$ and $0 < \varepsilon < 1/(2T)$. For $t \in [T^{-1}, T]$ and $s \in [\varepsilon, t - \varepsilon]$, we have by (3.10) and (3.11) that there is a constant $c_2 = c_2(d, \alpha, M, T, b)$ such that

$$|p_k^{a,b}(t-s, x, z)| \leq C_9(C_{12}C_{15}M_b(\sqrt{t}))^k q_{d,C_{10}/2}^a(t-s, x, z) \leq 2c_2(t-s)^{-d/2} \leq 2c_2\varepsilon^{-d/2}, \quad (3.12)$$

$$|\nabla_z p^a(s, z, y)| \leq C_{12}q_{d+1,3C_{10}/4}^a(s, z, y) \leq 2C_{12}s^{-(d+1)/2} \leq 2C_{12}\varepsilon^{-(d+1)/2}, \quad (3.13)$$

$$|p_k^{a,b}(t-s, x, z)| \leq 2c_2 \frac{1}{|x-z|^{d+\alpha}}, \quad \text{if } |x-z| \geq 1.$$

Then for $R \geq 1$,

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \int_\varepsilon^{t-\varepsilon} \int_{|x-z| \geq R} |p_k^{a,b}(t-s, x, z)| |b(z)| |\nabla_z p^a(s, z, y)| dz ds \\ &\leq \sup_{x \in \mathbb{R}^d} 4c_2 C_{12} \varepsilon^{-(d+1)/2} T \int_{|x-z| \geq R} \frac{|b(z)|}{|x-z|^{d+\alpha}} dz, \end{aligned}$$

which goes to zero as $R \rightarrow \infty$. On the other hand, since $x \mapsto p_k^{a,b}(t-s, x, z)$ is continuous by assumption and $p^a(t, x, y)$ is smooth, we have for any $r > 0$, by (3.12-3.13), the local integrability of b and the dominated convergence theorem,

$$(x, y) \mapsto \int_\varepsilon^{t-\varepsilon} \int_{|x-z| < R} p_k^{a,b}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds$$

is continuous on $B(0, r) \times B(x, r)$. Thus, we can conclude that

$$(t, x, y) \mapsto \int_\varepsilon^{t-\varepsilon} \int_{\mathbb{R}^d} |p_k^{a,b}(t-s, x, z)| |b(z)| |\nabla_z p^a(s, z, y)| dz ds \quad (3.14)$$

is jointly continuous on $[T^{-1}, T] \times B(0, r) \times B(0, r)$. Since r is arbitrary, (3.14) is jointly continuous on $[T^{-1}, T] \times \mathbb{R}^d \times \mathbb{R}^d$. On the other hand, by (3.13) and (3.10),

$$\begin{aligned}
& \sup_{t \in [1/T, T]} \sup_{x, y \in \mathbb{R}^d} \int_{t-\varepsilon}^t \int_{\mathbb{R}^d} |p_k^{a,b}(t-s, x, z)| |b(z)| |\nabla_z p^a(s, z, y)| dz ds \\
& \leq \sup_{t \in [1/T, T]} \sup_{x \in \mathbb{R}^d} \sup_{s \in [t-\varepsilon, t], z, y \in \mathbb{R}^d} |\nabla_z p^a(s, z, y)| \int_{t-\varepsilon}^t \int_{\mathbb{R}^d} |p_k^{a,b}(t-s, x, z)| |b(z)| dz ds \\
& \leq \sup_{s \in [1/(2T), T], z, y \in \mathbb{R}^d} |\nabla_z p^a(s, z, y)| \sup_{x \in \mathbb{R}^d} \int_0^\varepsilon \int_{\mathbb{R}^d} |p_k^{a,b}(s, x, z)| |b(z)| dz ds \\
& \leq 2C_{12}(2T)^{(d+1)/2} C_9 (C_{12} C_{15} M_b(\sqrt{\varepsilon}))^k \\
& \quad \cdot \sup_{x \in \mathbb{R}^d} \left(\int_0^\varepsilon \int_{\mathbb{R}^d} \sqrt{\varepsilon} |b(z)| \frac{g_{d, C_{10}/2}(s, x, z)}{s^{1/2}} dz ds + \int_0^\varepsilon \int_{|x-z|^2 \geq s} |b(z)| \frac{a^\alpha s}{|x-z|^{d+\alpha}} dz ds \right) \\
& \leq 2C_{12}(2T)^{(d+1)/2} C_9 (C_{12} C_{15} M_b(\sqrt{\varepsilon}))^k \\
& \quad \cdot \left(\sqrt{\varepsilon} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |b(z)| N^{C_{10}/2}(\varepsilon, x, z) dz + \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} a^\alpha |b(z)| \frac{\varepsilon^2 \wedge |x-z|^4}{|x-z|^{d+\alpha}} dz \right) \\
& \leq 2C_{12}(2T)^{(d+1)/2} C_9 (C_{12} C_{15} M_b(\sqrt{\varepsilon}))^k \left(c_3 C_{13} \sqrt{\varepsilon} M_b(\sqrt{\varepsilon}) + a^\alpha \varepsilon^{(3-\alpha)/2} C_{13} M_b(\sqrt{\varepsilon}) \right),
\end{aligned}$$

which goes to zero as $\varepsilon \rightarrow 0$. Similarly, by (3.12),

$$\begin{aligned}
& \sup_{t \in [1/T, T]} \sup_{x, y \in \mathbb{R}^d} \int_0^\varepsilon \int_{\mathbb{R}^d} |p_k^{a,b}(t-s, x, z)| |b(z)| |\nabla_z p^a(s, z, y)| dz ds \\
& \leq 2c_2 (2T)^{d/2} C_9 (C_{12} C_{15} M_b(\sqrt{\varepsilon}))^k \left(c_3 C_{13} M_b(\sqrt{\varepsilon}) + a^\alpha \varepsilon^{1-\alpha/2} C_{13} M_b(\sqrt{\varepsilon}) \right),
\end{aligned}$$

which goes to zero as $\varepsilon \rightarrow 0$. Therefore,

$$p_{k+1}^{a,b}(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_k^{a,b}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds$$

is jointly continuous on $[T^{-1}, T] \times \mathbb{R}^d \times \mathbb{R}^d$ for every $T > 0$. This completes the proof. \square

Lemma 3.4. Suppose $M > 0$. There are two positive constants $t_*(d, \alpha, M, b) > 0$ depending on b only via the rate at which $M_b(r)$ goes to zero and $C_{16} = C_{16}(d, \alpha, M) > 0$ such that for all $t \in (0, t_*]$ and $x, y \in \mathbb{R}^d$,

$$\left| \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) \right| \leq \sum_{k=0}^{\infty} |p_k^{a,b}(t, x, y)| \leq C_{16} q_{d, C_{10}/2}^a(t, x, y). \quad (3.15)$$

Moreover, for all $|x-y|^2 < t \leq t_*$,

$$\sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) \geq C_{16}^{-1} t^{-d/2}. \quad (3.16)$$

Proof. Let C_9, C_{12}, C_{15} be the constants in (3.10) with $T = 1$ and C_7, C_8 be the constants in Theorem 2.2 with $T = 1$. Since $b \in \mathbb{K}_{d,1}$, there is a constant $0 < t_* < 1$ such that for all $t \in (0, t_*]$

$$C_{12} C_{15} M_b(\sqrt{t}) \leq \frac{1}{2} \wedge \frac{C_7 e^{-C_8}}{8C_9},$$

and so by (3.10) with $T = 1$,

$$\sum_{k=1}^{\infty} |p_k^{a,b}(t, x, y)| \leq C_9 \frac{C_{12} C_{15} M_b(\sqrt{t})}{1 - C_{12} C_{15} M_b(\sqrt{t})} q_{d, C_{10}/2}^a(t, x, y)$$

$$\leq 2C_9C_{12}C_{15}M_b(\sqrt{t})q_{d,C_{10}/2}^a(t, x, y), \quad x, y \in \mathbb{R}^d. \quad (3.17)$$

Thus, by Theorem 2.1 with $T = 1$ and (3.17), we have for all $(t, x, y) \in (0, t_*] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\left| \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) \right| \leq \sum_{k=0}^{\infty} |p_k^{a,b}(t, x, y)| \leq 2C_9q_{d,C_{10}/2}^a(t, x, y),$$

which gives (3.15). On the other hand, if $|x - y|^2 < t \leq t_*$, then

$$p^a(t, x, y) \geq C_7e^{-C_8}t^{-d/2} \text{ and } q_{d,C_{10}/2}^a(t, x, y) \leq 2t^{-d/2}.$$

Thus, by (3.17) again, we have for $(t, x, y) \in (0, t_*] \times \mathbb{R}^d \times \mathbb{R}^d$ with $|x - y|^2 \leq t$,

$$\begin{aligned} \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) &\geq p^a(t, x, y) - \sum_{k=1}^{\infty} |p_k^{a,b}(t, x, y)| \geq C_7e^{-C_8}t^{-d/2} - \frac{C_7e^{-C_8}}{2}t^{-d/2} \\ &= \frac{C_7e^{-C_8}}{2}t^{-d/2}. \end{aligned}$$

□

In the remainder of this paper, we fix t_* . By Lemma 3.4, the series $\sum_{k=0}^{\infty} p_k^{a,b}(t, x, y)$ absolutely converges on $(0, t_*] \times \mathbb{R}^d \times \mathbb{R}^d$. For every $a \in (0, M]$, define

$$p^{a,b}(t, x, y) = \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y), \quad 0 < t \leq t_* \text{ and } x, y \in \mathbb{R}^d. \quad (3.18)$$

Lemma 3.5. *Suppose $M > 0$. For every $a \in (0, M]$, $p^{a,b}(t, x, y)$ is jointly continuous on $(0, t_*] \times \mathbb{R}^d \times \mathbb{R}^d$.*

Proof. For any $0 < t_1 < t_*$, we have

$$\sup_{[t_1, t_*] \times \mathbb{R}^d \times \mathbb{R}^d} q_{d,C_{10}/2}^a(t, x, y) \leq 2t_1^{-d/2} < \infty.$$

By Lemma 3.4 and inequality (3.10), the series $\sum_{k=0}^{\infty} p_k^{a,b}(t, x, y)$ converges uniformly on $[t_1, t_*] \times \mathbb{R}^d \times \mathbb{R}^d$. Since t_1 is arbitrary, the result follows from Lemma 3.3. □

Theorem 3.6. *Suppose $M > 0$. For every $a \in (0, M]$, $0 < s, t \leq t_*$ with $s + t \leq t_*$ and $x, y \in \mathbb{R}^d$, we have*

$$p^{a,b}(t + s, x, y) = \int_{\mathbb{R}^d} p^{a,b}(t, x, z) p^{a,b}(s, z, y) dz. \quad (3.19)$$

Proof. Note that for $s, t > 0$ with $s + t \leq t_*$,

$$\begin{aligned} p^{a,b}(t, x, z) p^{a,b}(s, z, y) &= \left(\sum_{m=0}^{\infty} p_m^{a,b}(t, x, z) \right) \left(\sum_{k=0}^{\infty} p_k^{a,b}(s, z, y) \right) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^k p_m^{a,b}(t, x, z) p_{k-m}^{a,b}(s, z, y). \end{aligned}$$

So it suffices to prove that for any $k \geq 0$,

$$p_k^{a,b}(t + s, x, y) = \sum_{m=0}^k \int_{\mathbb{R}^d} p_m^{a,b}(t, x, z) p_{k-m}^{a,b}(s, z, y) dz, \quad (3.20)$$

which will be done inductively. When $k = 0$, (3.20) is clearly true since $p_0^{a,b}(t, x, y) = p^a(t, x, y)$. Suppose (3.20) holds when $k = l$ and we have

$$\begin{aligned}
p_{l+1}^{a,b}(t+s, x, y) &= \int_0^{t+s} \int_{\mathbb{R}^d} p_l^{a,b}(t+s-\tau, x, w) b(w) \nabla_w p_0^{a,b}(\tau, w, y) dw d\tau \\
&= \left(\int_0^s + \int_s^{t+s} \right) \int_{\mathbb{R}^d} p_l^{a,b}(t+s-\tau, x, w) b(w) \nabla_w p_0^{a,b}(\tau, w, y) dw d\tau \\
&= \int_0^s \int_{\mathbb{R}^d} \sum_{m=0}^l \int_{\mathbb{R}^d} p_m^{a,b}(t, x, z) p_{l-m}^{a,b}(s-\tau, z, w) dz b(w) \nabla_w p_0^{a,b}(\tau, w, y) dw d\tau \\
&\quad + \int_s^{t+s} \int_{\mathbb{R}^d} p_l^{a,b}(t+s-\tau, x, w) b(w) \nabla_w \int_{\mathbb{R}^d} p_0^{a,b}(\tau-s, w, z) p_0^{a,b}(s, z, y) dz dw d\tau \\
&= \sum_{m=0}^l \int_{\mathbb{R}^d} p_m^{a,b}(t, x, z) \int_0^s \int_{\mathbb{R}^d} p_{l-m}^{a,b}(s-\tau, z, w) b(w) \nabla_w p_0^{a,b}(\tau, w, y) dw d\tau dz \\
&\quad + \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p_l^{a,b}(t-\tau, x, w) b(w) \nabla_w p_0^{a,b}(\tau, w, z) dw d\tau p_0^{a,b}(s, z, y) dz \\
&= \sum_{m=0}^l \int_{\mathbb{R}^d} p_m^{a,b}(t, x, z) p_{l+1-m}^{a,b}(s, z, y) dz + \int_{\mathbb{R}^d} p_{l+1}^{a,b}(t, x, z) p_0^{a,b}(s, z, y) dz,
\end{aligned}$$

where in the second to the last equality, we used Fubini's theorem since for $(t, x, y) \in (0, t_*] \times \mathbb{R}^d \times \mathbb{R}^d$ and any $m, l \in \mathbb{Z}_+$, by (3.9) and Lemma 3.2,

$$\begin{aligned}
&\int_0^s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |p_m^{a,b}(t, x, w)| |p_l^{a,b}(s-\tau, w, z)| |b(z)| |\nabla_z p^a(\tau, z, y)| dw dz d\tau \\
&= \int_{\mathbb{R}^d} |p_m^{a,b}(t, x, w)| \int_0^s \int_{\mathbb{R}^d} |p_l^{a,b}(s-\tau, w, z)| |b(z)| |\nabla_z p^a(\tau, z, y)| dz d\tau dw \\
&\leq \int_{\mathbb{R}^d} |p_m^{a,b}(t, x, w)| |p^{a,b}|_{l+1}(s, w, y) dw < \infty.
\end{aligned}$$

We have also used the fact that due to Theorem 2.1 and the dominated convergence theorem, $\nabla_z p^a(\tau, z, y) = \int_{\mathbb{R}^d} \nabla_z p^a(\tau-s, z, w) p^a(s, w, y) dw$. \square

In view of Theorem 3.6, the definition of $p^{a,b}(t, x, y)$ can be uniquely extended to all $t > 0$ so that (1.8) holds for all $s, t > 0$. Suppose $p^{a,b}(t, x, y)$ has been well defined on $(0, kt_*] \times \mathbb{R}^d \times \mathbb{R}^d$ for integer $k \geq 0$ and (1.8) holds for all $s, t > 0$ with $s+t \leq kt_*$. For $t \in (kt_*, (k+1)t_*]$, we define

$$p^{a,b}(t, x, y) = \int_{\mathbb{R}^d} p^{a,b}(kt_*, x, z) p^{a,b}(t-kt_*, z, y) dz, \quad x, y \in \mathbb{R}^d. \quad (3.21)$$

One can verify easily that the Chapman-Kolmogorov equation (1.8) holds for every $t, s > 0$ with $t+s \leq (k+1)t_*$. This proves that (1.8) holds for all $t, s > 0$.

Theorem 3.7. *Suppose $M > 0$. For every $a \in (0, M]$, $p^{a,b}(t, x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\int_{\mathbb{R}^d} p^{a,b}(t, x, y) dy = 1$ for every $t > 0$ and $x \in \mathbb{R}^d$.*

Proof. The continuity of $p^{a,b}(t, x, y)$ for all $t > 0$ follows from Lemma 3.5, (3.21) and the dominated convergence theorem.

It follows from (2.5) that $\int_{\mathbb{R}^d} \nabla_x p^a(t, x, y) dy = 0$ for all $t > 0$ and $x \in \mathbb{R}^d$. Thus for every $k \geq 1$, by Lemma 3.2, (1.7), (3.10) and Fubini's theorem,

$$\int_{\mathbb{R}^d} p_k^{a,b}(t, x, y) dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t p_{k-1}^{a,b}(t-s, x, z) b(z) \cdot \nabla_z p^a(s, z, y) ds dz dy$$

$$= \int_{\mathbb{R}^d} \int_0^t p_{k-1}^{a,b}(t-s, x, z) b(z) \cdot \int_{\mathbb{R}^d} \nabla_z p^a(s, z, y) dy ds dz = 0.$$

In view of (3.10) and the dominated convergence theorem, we have for all $t \in (0, t_*]$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^{a,b}(t, x, y) dy = \sum_{k=0}^{\infty} \int_{\mathbb{R}^d} p_k^{a,b}(t, x, y) dy = \int_{\mathbb{R}^d} p^a(t, x, y) dy = 1,$$

which extends to all $t > 0$ by (3.21). \square

For bounded measurable function f on \mathbb{R}^d , $t > 0$ and $x \in \mathbb{R}^d$, we define operator $P_t^{a,b}$

$$P_t^{a,b} f(x) = \int_{\mathbb{R}^d} p^{a,b}(t, x, y) f(y) dy.$$

It follows from (3.19) that $P_s^{a,b} P_t^{a,b} = P_{t+s}^{a,b}$.

The following theorem tells us that the generator of $\{P_t^{a,b}, t \geq 0\}$ is $\mathcal{L}^{a,b}$ in the weak sense. The proof is almost the same to part of the proof of [4, Theorem 1]. We give the details of the proof for completeness. For any compact set $K \subset \mathbb{R}^d$ and $r > 0$, let $K^r = \{y \in \mathbb{R}^d : \exists x \in K \text{ such that } |x - y| < r\}$ be the r -neighborhood of K .

Theorem 3.8. *Suppose $M > 0$. For every $a \in (0, M]$ and for all $f \in C_c^\infty(\mathbb{R}^d)$, $g \in C_\infty(\mathbb{R}^d)$,*

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^d} \frac{P_t^{a,b} f(x) - f(x)}{t} g(x) dx = \int_{\mathbb{R}^d} \mathcal{L}^{a,b} f(x) g(x) dx.$$

Proof. Note that for all $t \in (0, t_*]$,

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{P_t^{a,b} f(x) - f(x)}{t} g(x) dx &= \frac{1}{t} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_0^{a,b}(t, x, y) f(y) dy - f(x) \right) g(x) dx \\ &\quad + \frac{1}{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(p_1^{a,b}(t, x, y) + \sum_{k=2}^{\infty} p_k^{a,b}(t, x, y) \right) f(y) g(x) dy dx. \end{aligned}$$

Since $p_0^{a,b}(t, x, y) = p^a(t, x, y)$ we have

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_0^{a,b}(t, x, y) f(y) dy - f(x) \right) g(x) dx = \int_{\mathbb{R}^d} \left(\Delta + a^\alpha \Delta^{\alpha/2} \right) f(x) g(x) dx.$$

For $t \in (0, t_*]$, let $I(t) = t^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_1^{a,b}(t, x, y) f(y) g(x) dx$. We claim that $I(t)$ converges to $\int_{\mathbb{R}^d} (b(x) \cdot \nabla f(x)) g(x) dx$ as $t \rightarrow 0$. By (1.7), Fubini's theorem and integration by parts,

$$I(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \frac{1}{t} p^a(t-s, x, z) p^a(s, z, y) ds (b(z) \cdot \nabla f(y)) g(x) dz dy dx.$$

Since $g(x) \nabla f(y)$ is uniformly continuous and bounded, for every $\varepsilon > 0$, there is $\delta > 0$ so that $|g(x) \nabla f(y) - g(z) \nabla f(w)| < \varepsilon$ for $|x - z| < \delta$ and $|y - w| < \delta$. Let $M_0 = \sup_{x, y \in \mathbb{R}^d} |g(x) \nabla f(y)|$, and K be the support of ∇f . Recall that K^1 denotes the 1-neighborhood of K . Clearly

$$\begin{aligned} &|I(t) - \int_{\mathbb{R}^d} (b(z) \cdot \nabla f(z)) g(z) dz| \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \frac{1}{t} p^a(t-s, x, z) p^a(s, z, y) ds |b(z)| |g(x) \nabla f(y) - g(z) \nabla f(z)| dx dy dz \\ &= \left(\int_{(K^1)^c} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^t + \int_{K^1} \int_{(B(z, \delta) \times B(z, \delta))^c} \int_0^t + \int_{K^1} \int_{B(z, \delta) \times B(z, \delta)} \int_0^t \right) \cdots ds dx dy dz \end{aligned}$$

$$=: J_1 + J_2 + J_3.$$

We estimate J_1, J_2 and J_3 separately. Note that if $x \in K$ and $z \in (K^1)^c$, then $|x - z| \geq 1$ and so by Theorem 2.1, for $x, y \in \mathbb{R}^d$ and $0 < s < t$,

$$p^a(t - s, x, z) \leq C_9 q_{d, C_{10}}^a(t - s, x, z) \leq C_9 c_1 \frac{t - s}{|x - z|^{d+\alpha}}.$$

where c_1 is a positive constant depending only on d, α, M . Thus,

$$\begin{aligned} J_1 &= \int_{(K^1)^c} \int_{K \times \mathbb{R}^d} \int_0^t \frac{1}{t} p^a(t - s, x, z) p^a(s, z, y) ds |b(z)| |g(x) \nabla f(y)| dx dy dz \\ &\leq M_0 \int_{(K^1)^c} \int_K \int_0^t \left(\int_{\mathbb{R}^d} p^a(s, z, y) dy \right) \frac{1}{t} p^a(t - s, x, z) |b(z)| ds dx dz \\ &\leq M_0 C_9 c_1 \int_{(K^1)^c} \int_K \int_0^t \frac{1}{t} \frac{t - s}{|x - z|^{d+\alpha}} |b(z)| ds dx dz \\ &\leq t M_0 C_9 c_1 |K| \sup_{x \in \mathbb{R}^d} \int_{|x - z| \geq 1} \frac{|b(z)|}{|x - z|^{d+\alpha}} dz \\ &\leq t M_0 C_9 c_1 |K| \sup_{x \in \mathbb{R}^d} H_b^{(3-\alpha)/2}(1, x), \end{aligned}$$

which tends to zero as $t \rightarrow 0$. Similarly, if $(x, y) \in (B(z, \delta) \times B(z, \delta))^c$, then $|x - z| \geq \delta$ or $|y - z| \geq \delta$. Since b is locally integrable, we have

$$J_2 \leq 2t M_0 C_9 c_1 \int_{K^1} \int_{|x - z| \geq \delta} |b(z)| \frac{1}{|x - z|^{d+\alpha}} dx dz \leq 2t M_0 C_9 c_1 \delta^{-d-2} \int_{K^1} |b(z)| dz \rightarrow 0$$

as $t \rightarrow 0$.

$$J_3 \leq \varepsilon \int_{K^1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \frac{1}{t} p^a(t - s, x, z) p^a(s, z, y) ds |b(z)| dx dy dz \leq \varepsilon \int_{K^1} |b(z)| dz.$$

Since ε is arbitrary, we have $\lim_{t \rightarrow 0} I(t) = \int_{\mathbb{R}^d} (b(z) \cdot \nabla f(z)) g(z) dz$.

By (1.7), Lemma 3.4 and the dominated convergence theorem, we have

$$\sum_{k=2}^{\infty} p_k^{a,b}(t, x, y) = \int_{\mathbb{R}^d} \int_0^t \left(\sum_{k=1}^{\infty} p_k^{a,b}(t - s, x, z) \right) b(z) \cdot \nabla_z p^a(s, z, y) ds dz.$$

Similar to the estimate of $I(t)$, by Fubini's theorem, integration by parts and (3.17), we have for all $t \in (0, t_*)]$

$$\begin{aligned} &\frac{1}{t} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\sum_{k=2}^{\infty} p_k^{a,b}(t, x, y) \right) f(y) g(x) dy dx \right| \\ &= \frac{1}{t} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \left(\sum_{k=1}^{\infty} p_k^{a,b}(t - s, x, z) \right) p^a(s, z, y) (b(z) \cdot \nabla f(y)) g(x) ds dz dy dx \right| \\ &\leq \frac{2C_9 C_{12} C_{15} M_b(\sqrt{t})}{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t q_{d, C_{10}}^a(t - s, x, z) p^a(s, z, y) |b(z)| |g(x) \nabla f(y)| ds dz dy dx \\ &\leq 2C_9 C_{12} C_{15} M_b(\sqrt{t}) \left(\frac{2C_8}{C_{10}} \right)^{d/2} C_7^{-1} \\ &\quad \times \frac{1}{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t p^a\left(\frac{2C_8}{C_{10}}(t - s), x, z\right) p^a(s, z, y) |b(z)| |g(x) \nabla f(y)| ds dz dy dx, \end{aligned}$$

which goes to zero as $t \rightarrow 0$. This completes the proof. \square

4 Uniqueness and Positivity

Theorem 4.1. *Suppose $M > 0$. There are constants $C_{17} = C_{17}(d, \alpha, M)$, $C_{18} = C_{18}(d, \alpha, M, b)$ such that for all $a \in (0, M]$,*

$$\left| p^{a,b}(t, x, y) \right| \leq C_{17} e^{C_{18}t} p^a(2C_8 t / C_{10}, x, y), \quad t > 0 \text{ and } x, y \in \mathbb{R}^d. \quad (4.1)$$

Consequently, for any $T > 0$, there is a constant $C_{19} = C_{19}(d, \alpha, M, T)$ such that

$$\left| p^{a,b}(t, x, y) \right| \leq C_{19} e^{C_{18}t} q_{d, C_{10}^2/(2C_8)}^a(t, x, y), \quad t \in (0, T] \text{ and } x, y \in \mathbb{R}^d.$$

Proof. Note that by the expression of $q_{d, C_{10}^2/(2C_8)}^a(t, x, y)$ and the lower bound of $p^a(t, x, y)$ in Theorem 2.1 with $T = 1$,

$$q_{d, C_{10}^2/(2C_8)}^a(t, x, y) \leq \left(\frac{2C_8}{C_{10}} \right)^{d/2} q_{d, C_8}^a(2C_8 t / C_{10}, x, y) \leq \left(\frac{2C_8}{C_{10}} \right)^{d/2} C_7^{-1} p^a(2C_8 t / C_{10}, x, y). \quad (4.2)$$

Recall that t_* is the constant in Lemma 3.4. If $t < t_*$, by (3.15) and Theorem 2.1,

$$\left| p^{a,b}(t, x, y) \right| \leq C_{16} q_{d, C_{10}^2/(2C_8)}^a(t, x, y) \leq c_1 p^a(2C_8 t / C_{10}, x, y),$$

where $c_1 = \frac{C_{16}(2C_8)^{d/2}}{C_7 C_{10}^{d/2}}$ depends only on d, α, M . It remains to consider the case $t > t_*$. Let $k = \lfloor t/t_* \rfloor + 1$, then $t/k \in (0, t_*)$. Combining (4.2), (1.8) and (3.15), we have

$$\begin{aligned} \left| p^{a,b}(t, x, y) \right| &\leq \int_{\mathbb{R}^{d(k-1)}} c_1^k p^a\left(\frac{2C_8}{C_{10}} \frac{t}{k}, x, x_1\right) \cdots p^a\left(\frac{2C_8}{C_{10}} \frac{t}{k}, x_{k-1}, y\right) dx_1 \cdots dx_{k-1} \\ &= c_1^k p^a(2C_8 t / C_{10}, x, y) \\ &\leq c_1 c_1^{\frac{t}{t_*}} p^a(2C_8 t / C_{10}, x, y), \end{aligned}$$

which gives the first conclusion with $C_{17} = c_1$ and $C_{18} = \frac{1}{t_*} \ln c_1$. Furthermore, by the upper bound of $p^a(t, x, y)$ in Theorem 2.1, for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$

$$p^a(2C_8 t / C_{10}, x, y) \leq C_9 q_{d, C_{10}^2/(2C_8)}^a(2C_8 t / C_{10}, x, y) \leq \frac{2C_8 C_9}{C_{10}} q_{d, C_{10}^2/(2C_8)}^a(t, x, y).$$

Combining the last two displays, we finish the proof by setting $C_{19} = c_1((2C_8 C_9 / C_{10}) \vee 1)$. \square

Theorem 4.2. *Suppose that $M > 0$ and $b \in \mathbb{K}_{d,1}$. For every $a \in (0, M]$, $p^{a,b}(t, x, y)$ satisfies (1.6) for all $t > 0$ and $x, y \in \mathbb{R}^d$.*

Proof. Recall that t^* is the constant in Lemma 3.4. We first prove that $p^{a,b}(t, x, y)$ satisfies (1.6) for all $t \in (0, t^*]$ and $x, y \in \mathbb{R}^d$. Indeed, by (3.18), (3.17), Theorem 2.2, (3.2) and the dominated convergence theorem, we have for all $t \in (0, t^*]$,

$$\begin{aligned} p^{a,b}(t, x, y) &= \sum_{n=0}^{\infty} p_n^{a,b}(t, x, y) \\ &= p^a(t, x, y) + \sum_{n=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} p_{n-1}^{a,b}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds \\ &= p^a(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} p_{n-1}^{a,b}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds \\ &= p^a(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p^{a,b}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds. \end{aligned}$$

Now, we use induction in k to prove (1.6) for all $t > 0$. Suppose that (1.6) is true for $t \in (0, 2^k t^*](k \geq 0)$ and for all $x, y \in \mathbb{R}^d$. We will prove (1.6) is true for $t \in (2^k t^*, 2^{k+1} t^*]$. Setting $s = t/2 \in (2^{k-1} t^*, 2^k t^*]$, by (1.8), Theorem 4.1, (3.2) and Fubini's theorem, we have

$$\begin{aligned}
p^{a,b}(t, x, y) &= \int_{\mathbb{R}^d} p^{a,b}(s, x, z) p^{a,b}(s, z, y) dz \\
&= \int_{\mathbb{R}^d} p^{a,b}(s, x, z) \left(p^a(s, z, y) + \int_0^s \int_{\mathbb{R}^d} p^{a,b}(s-r, z, w) b(w) \nabla_w p^a(r, w, y) dw dr \right) dz \\
&= \int_{\mathbb{R}^d} p^a(s, x, z) p^a(s, z, y) dz \\
&\quad + \int_{\mathbb{R}^d} \left(\int_0^s \int_{\mathbb{R}^d} p^{a,b}(s-r, x, u) b(u) \nabla_u p^a(r, u, z) du dr \right) p^a(s, z, y) dz \\
&\quad + \int_{\mathbb{R}^d} p^{a,b}(s, x, z) \left(\int_0^s \int_{\mathbb{R}^d} p^{a,b}(s-r, z, w) b(w) \nabla_w p^a(r, w, y) dw dr \right) dz \\
&= p^a(t, x, y) + \int_0^s \int_{\mathbb{R}^d} p^{a,b}(s-r, x, u) b(u) \left(\int_{\mathbb{R}^d} \nabla_u p^a(r, u, z) p^a(s, z, y) dz \right) du dr \\
&\quad + \int_0^s \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p^{a,b}(s, x, z) p^{a,b}(s-r, z, w) dz \right) b(w) \nabla_w p^a(r, w, y) dw dr \\
&= p^a(t, x, y) + \int_0^s \int_{\mathbb{R}^d} p^{a,b}(s-r, x, u) b(u) \nabla_u p^a(r+s, u, y) du dr \\
&\quad + \int_0^s \int_{\mathbb{R}^d} p^{a,b}(2s-r, x, w) b(w) \nabla_w p^a(r, w, y) dw dr \\
&= p^a(t, x, y) + \int_s^{2s} \int_{\mathbb{R}^d} p^{a,b}(2s-r, x, u) b(u) \nabla_u p^a(r, u, y) du dr \\
&\quad + \int_0^s \int_{\mathbb{R}^d} p^{a,b}(2s-r, x, w) b(w) \nabla_w p^a(r, w, y) dw dr \\
&= p^a(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p^{a,b}(t-r, x, z) b(z) \nabla_z p^a(r, z, y) dz dr,
\end{aligned}$$

where in the forth equality, we can change the order of integral and ∇ , since for any $t_1, t_2 \in (0, \infty)$ and $x, y \in \mathbb{R}^d$,

$$\nabla_x p^a(t_1 + t_2, x, y) = \int_{\mathbb{R}^d} \nabla_x p^a(t_1, x, z) p^a(t_2, z, y) dz,$$

which can be proved by Theorem 2.2 and the dominated convergence theorem. \square

Theorem 4.3. Suppose that $M > 0$ and $b \in \mathbb{K}_{d,1}$. For every $a \in (0, M]$, $p^{a,b}(t, x, y)$ is the unique continuous heat kernel that satisfies the Chapman-Kolmogorov equation (1.8) on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, Duhamel's formula (1.6) on $(0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d$ for some constant $t_0 > 0$ and that for some $c_1 > 0$,

$$\left| p^{a,b}(t, x, y) \right| \leq c_1 p^a(t, x, y) \quad \text{for } t \in (0, t_0] \text{ and } x, y \in \mathbb{R}^d. \quad (4.3)$$

Proof. Suppose that $\bar{p}(t, x, y)$ is any continuous heat kernel that satisfies Duhamel's formula (1.6) and (4.3) for $(t, x, y) \in (0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d$. Without loss of generality, we may and do assume that $t_0 \leq t^*$. Firstly, let $R_1(t, x, y) = \int_0^t \int_{\mathbb{R}^d} \bar{p}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds$ and

$$R_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} R_{n-1}(t-s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds, \quad n \geq 2.$$

Similar to the arguments that lead to (3.10), by (4.3), we can recursively verify that $R_n(t, x, y)$ is well defined. Furthermore, we have the upper bound of $|R_n(t, x, y)|$:

$$|R_n(t, x, y)| \leq c_1 (C_{12} C_{15} M_b(\sqrt{t}))^n q_{d, C_{10}/2}^a(t, x, y).$$

On the other hand, using Duhamel's formula (1.6) inductively, we have for every $n \geq 1$,

$$\bar{p}(t, x, y) = \sum_{j=0}^{n-1} p_j^{a,b}(t, x, y) + R_n(t, x, y),$$

where $p_j^{a,b}(t, x, y)$ is defined by (1.7). Note that for all $(t, x, y) \in (0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d$, by the proof of Lemma 3.4, $C_{12} C_{15} M_b(\sqrt{t}) \leq 1/2$ and so

$$|R_n(t, x, y)| \leq c_1 2^{-n} q_{d, C_{10}/2}^a(t, x, y) < \infty,$$

which goes to zero as $n \rightarrow \infty$. Thus, we have

$$\bar{p}(t, x, y) = \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) = p^{a,b}(t, x, y), \quad \text{for all } (t, x, y) \in (0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d.$$

Since both \bar{p} and $p^{a,b}$ satisfy the Chapman-Kolmogorov equation (1.8) on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, we have $\bar{p} = p^{a,b}$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. \square

Unlike that in [4], it is not easy to show the positivity of $p^{a,b}(t, x, y)$ directly from its construction. We show $p^{a,b}(t, x, y) \geq 0$ by adopting the approach from [10], using Hille-Yosida-Ray theorem when b is bounded and continuous and then using approximation for general b .

Lemma 4.4. *Suppose $M > 0$. For every $a \in (0, M]$ and every $t > 0$, $P_t^{a,b}$ maps bounded functions to continuous functions. Furthermore, $\{P_t^{a,b}, t \geq 0\}$ is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$.*

Proof. By Theorem 4.1 and Theorem 3.7, one can easily verify that $P_t^{a,b}$ maps bounded functions to continuous functions for every $t > 0$. For every $f \in C_\infty(\mathbb{R}^d)$ and $t > 0$, by Theorem 4.1,

$$\begin{aligned} \lim_{|x| \rightarrow \infty} |P_t^{a,b} f(x)| &\leq \lim_{|x| \rightarrow \infty} \int_{\mathbb{R}^d} C_{19} e^{C_{18}t} q_{d, C_{10}^2/(2C_8)}^a(t, x, y) f(y) dy \\ &\leq \lim_{|x| \rightarrow \infty} \int_{\mathbb{R}^d} C_{19} e^{C_{18}t} q_{d, C_{10}^2/(2C_8)}^a(t, 0, y) f(x+y) dy = 0, \end{aligned}$$

which shows $P_t^{a,b} f \in C_\infty(\mathbb{R}^d)$. Moreover, since f is uniformly continuous on \mathbb{R}^d , for every $\varepsilon > 0$, there is a constant $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ for all $x, y \in \mathbb{R}^d$ with $|x - y| \leq \delta$. And so by (3.11),

$$\begin{aligned} &\limsup_{t \rightarrow 0} \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq \delta} |p^{a,b}(s, x, y)| dy \\ &\leq \limsup_{t \rightarrow 0} \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq \delta} C_{19} e^{C_{18}s} q_{d, C_{10}^2/(2C_8)}^a(s, x, y) dy \\ &\leq \limsup_{t \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq \delta} C_{19} e^{C_{18}t} c_1 \left(\frac{t}{|x-y|^{d+2}} + \frac{t}{|x-y|^{d+\alpha}} \right) dy = 0, \end{aligned}$$

where c_1 is some positive constant depending only on d, α, M . Thus, we have

$$\limsup_{t \rightarrow 0} \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} |P_s^{a,b} f(x) - f(x)| = \limsup_{t \rightarrow 0} \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^{a,b}(s, x, y) f(y) dy - f(x) \right|$$

$$\begin{aligned}
&\leq \limsup_{t \rightarrow 0} \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \delta} |p^{a,b}(s, x, y)| |f(x) - f(y)| dy \\
&\leq \limsup_{t \rightarrow 0} \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \delta} C_{17} e^{C_{18}t} p^a(2C_8^2 s / C_{10}^2, x, y) |f(x) - f(y)| dy \\
&\leq \varepsilon C_{17},
\end{aligned}$$

which shows that $\lim_{t \rightarrow 0} \|P_t^{a,b} f - f\|_\infty = 0$. \square

Lemma 4.5. *Suppose $M > 0$ and the function b is bounded and continuous on \mathbb{R}^d . Then, for every $a \in (0, M]$,*

$$p^{a,b}(t, x, y) \geq 0, \quad t > 0 \text{ and } x, y \in \mathbb{R}^d.$$

Proof. Denote the Feller generator of $\{P_t^{a,b}, t \geq 0\}$ in $C_\infty(\mathbb{R}^d)$ by $\widehat{\mathcal{L}}^{a,b}$, which is a closed operator. For every $f \in C_c^2(\mathbb{R}^d)$, since b is continuous, it is easy to see that $\mathcal{L}^{a,b} f \in C_\infty(\mathbb{R}^d)$. Similar to Theorem 3.8, we claim that $(P_t^{a,b} f - f)/t$ uniformly converges to $\mathcal{L}^{a,b} f$ as $t \rightarrow 0$. Indeed, for any $t \in (0, t_*]$,

$$\begin{aligned}
&\|(P_t^{a,b} f - f)/t - \mathcal{L}^{a,b} f\|_\infty \\
&= \sup_{x \in \mathbb{R}^d} \left| \frac{1}{t} \left(\int_{\mathbb{R}^d} \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) f(y) dy - f(x) \right) - \left(\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla \right) f(x) \right| \\
&\leq \sup_{x \in \mathbb{R}^d} \left| \frac{1}{t} \left(\int_{\mathbb{R}^d} p_0^{a,b}(t, x, y) f(y) dy - f(x) \right) - \left(\Delta + a^\alpha \Delta^{\alpha/2} \right) f(x) \right| \\
&\quad + \sup_{x \in \mathbb{R}^d} \left| \frac{1}{t} \int_{\mathbb{R}^d} p_1^{a,b}(t, x, y) f(y) dy - b(x) \nabla f(x) \right| \\
&\quad + \sup_{x \in \mathbb{R}^d} \left| \frac{1}{t} \int_{\mathbb{R}^d} \sum_{k=2}^{\infty} p_k^{a,b}(t, x, y) f(y) dy \right| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

It follows that I_1 goes to zero as $t \rightarrow 0$ since $\Delta + a^\alpha \Delta^{\alpha/2}$ is the generator of Z^a . We next treat I_2 as we did with I in the proof of Theorem 3.8. Let $M_0 = \sup_{x \in \mathbb{R}^d} |b(x) \nabla f(x)|$. Since $f \in C_c^2(\mathbb{R}^d)$, for any $\varepsilon > 0$, there is constant $\delta > 0$ such that $|b(z) \nabla f(y) - b(x) \nabla f(x)| < \varepsilon$ for all $x \in \mathbb{R}^d$ and $(z, y) \in B(x, \delta) \times B(x, \delta)$. On the other hand, for $(z, y) \in (B(x, \delta) \times B(x, \delta))^c$, either $|z - x| \geq \delta$ or $|y - x| \geq \delta$. Hence by (3.11),

$$\begin{aligned}
I_2 &\leq \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{t} p^a(t-s, x, z) p^a(s, z, y) |b(z) \nabla_y f(y) - b(x) \nabla_x f(x)| dz dy ds \\
&\leq \sup_{x \in \mathbb{R}^d} \int_{B(x, \delta) \times B(x, \delta)} \int_0^t \frac{1}{t} p^a(t-s, x, z) p^a(s, z, y) |b(z) \nabla_y f(y) - b(x) \nabla_x f(x)| ds dz dy \\
&\quad + \sup_{x \in \mathbb{R}^d} \int_{(B(x, \delta) \times B(x, \delta))^c} \int_0^t \dots ds dz dy \\
&\leq \varepsilon + 2M_0 \sup_{x \in \mathbb{R}^d} \int_{|x-z| \geq \delta} \int_0^t c_1 \left(\frac{t-s}{|x-z|^{d+2}} + \frac{t-s}{|x-z|^{d+\alpha}} \right) ds dz \\
&\leq \varepsilon + M_0 c_1 \omega_d (\delta^{-2}/2 + \delta^\alpha/\alpha) t.
\end{aligned}$$

where c_1 is some positive constant depending only on d, α, M . Since ε is arbitrary, I_2 goes to zero as $t \rightarrow 0$. Similar to I_2 , we can prove that I_3 goes to zero as $t \rightarrow 0$. Thus we have

$$C_c^2(\mathbb{R}^d) \subset D(\widehat{\mathcal{L}}^{a,b}) \text{ and } \widehat{\mathcal{L}}^{a,b} f = \mathcal{L}^{a,b} f \text{ for all } f \in C_c^2(\mathbb{R}^d). \quad (4.4)$$

On the other hand, for $\lambda > C_{18}$, by Theorem 4.1,

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \int_0^\infty e^{-\lambda t} |P_t^{a,b} f(x)| dt &\leq \sup_{x \in \mathbb{R}^d} \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^d} |p^{a,b}(t, x, y)| |f(y)| dy dt \\ &\leq \|f\|_\infty \int_0^\infty C_{17} e^{-(\lambda - C_{18})t} dt = c_\lambda \|f\|_\infty, \end{aligned} \quad (4.5)$$

where $c_\lambda = C_{17}/(\lambda - C_{18})$. Consider the strongly continuous semigroup $\{e^{-C_{18}t} P_t^{a,b}, t \geq 0\}$ with its generator $\widehat{\mathcal{L}}^{a,b} - C_{18}$. By (4.5), the resolvent set $\rho(\widehat{\mathcal{L}}^{a,b} - C_{18})$ of $\widehat{\mathcal{L}}^{a,b} - C_{18}$ contains $(0, \infty)$. Moreover, $\widehat{\mathcal{L}}^{a,b} - C_{18}$ satisfies the positive maximum principle in view of (4.4) and [1, Theorem 3.5.3]. Therefore, $\{e^{-C_{18}t} P_t^{a,b}, t \geq 0\}$ is a positivity preserving semigroup on $C_\infty(\mathbb{R}^d)$ by Hille-Yosida-Ray theorem (see [1, Theorem 3.5.1]). Since $\{e^{-C_{18}t} P_t^{a,b}, t \geq 0\}$ has a continuous kernel $e^{-C_{18}t} p^{a,b}(t, x, y)$, we have $p^{a,b}(t, x, y) \geq 0$ for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. \square

In the rest of this section, we show by an approximation argument that Lemma 4.5 continues to hold for $b \in \mathbb{K}_{d,1}$. Let φ be a non-negative function in $C_c^\infty(\mathbb{R}^d)$ with $\text{supp}(\varphi) \subset B(0, 1)$ and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. For $n \geq 1$, define $\varphi_n(x) := n^d \varphi(nx)$ and

$$b_n(x) = \int_{\mathbb{R}^d} \varphi_n(x - y) b(y) dy, \quad x \in \mathbb{R}^d.$$

For any compact set $K \subset \mathbb{R}^d$ and $r > 0$, recall that K^r is the r -neighborhood of K . For any $0 \leq r_1 \leq r_2 \leq +\infty$ and $\beta \geq 0$, we have

$$\begin{aligned} \sup_{x \in K} \int_{|x-y| \in [r_1, r_2]} \frac{|b_n(y)|}{|x-y|^{d-1+2\beta}} dy &\leq \sup_{x \in K} \int_{|x-y| \in [r_1, r_2]} \int_{\mathbb{R}^d} \frac{\varphi_n(y-z) |b(z)|}{|x-y|^{d-1+2\beta}} dz dy \\ &= \sup_{x \in K} \int_{|x-y| \in [r_1, r_2]} \int_{|z| < 1/n} \frac{\varphi_n(z) |b(y-z)|}{|x-y|^{d-1+2\beta}} dz dy \\ &= \sup_{x \in K} \int_{|z| < 1/n} \varphi_n(z) \int_{|x-z-y| \in [r_1, r_2]} \frac{|b(y)|}{|x-z-y|^{d-1+2\beta}} dy dz \\ &\leq \int_{|z| < 1/n} \varphi_n(z) \sup_{x \in K^1} \int_{|x-y| \in [r_1, r_2]} \frac{|b(y)|}{|x-y|^{d-1+2\beta}} dy dz \\ &= \sup_{x \in K^1} \int_{|x-y| \in [r_1, r_2]} \frac{|b(y)|}{|x-y|^{d-1+2\beta}} dy. \end{aligned} \quad (4.6)$$

In particular, for every $r > 0$ and $n \geq 1$, by setting $r_1 = 0, r_2 = r$ and $\beta = 0$, we have

$$M_{b_n}(r) \leq M_b(r). \quad (4.7)$$

Recall that $\gamma = (1 + \alpha \wedge 1)/2$.

Lemma 4.6. $H_{b-b_n}^\gamma(t, x)$ converges to 0 uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}^d$ as $n \rightarrow \infty$.

Proof. Let $[t_0, T_0] \times K \subset (0, +\infty) \times \mathbb{R}^d$ be an arbitrary compact set. Then, we have

$$\begin{aligned} \sup_{(t,x) \in [t_0, T_0] \times K} H_{b-b_n}^\gamma(t, x) &\leq \sup_{x \in K} \int_{\mathbb{R}^d} \left(\frac{1}{|x-y|^{d-1}} \wedge \frac{T_0^\gamma}{|x-y|^{d-1+2\gamma}} \right) |b(y) - b_n(y)| dy \\ &\leq \sup_{x \in K} \left(\int_{|x-y| < r} + \int_{r \leq |x-y| < R} + \int_{|x-y| \geq R} \right) \cdots dy \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where $0 < r < R < \infty$. Taking $r_1 = 0, r_2 = r$ and $\beta = 0$ in (4.6), we have

$$I_1 \leq 2 \sup_{x \in K^1} \int_{|x-y| < r} \frac{|b(y)|}{|x-y|^{d-1}} dy.$$

Since $b \in \mathbb{K}_{d,1}$, for any $\varepsilon > 0$, we can choose r small enough such that $I_1 \leq 2 \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$. Taking $r_1 = R, r_2 = \infty$ and $\beta = \gamma$ in (4.6), we have

$$I_3 \leq 2T_0^\gamma \sup_{x \in K^1} \int_{|x-y| \geq R} \frac{|b(y)|}{|x-y|^{d-1+2\gamma}} dy.$$

Fix a point $x_0 \in K^1$ and take $R > \text{diam}(K^1)$. Note that for $x \in K^1$ with $|x-y| \geq R$, we have

$$\begin{aligned} |x_0 - y| &\geq |x - y| - |x - x_0| \geq R - \text{diam}(K^1), \\ \frac{|x_0 - y|}{|x - y|} &\leq \frac{|x_0 - x| + |x - y|}{|x - y|} \leq \frac{\text{diam}(K^1)}{R} + 1 \leq 2, \end{aligned}$$

and so

$$I_3 \leq 2^{d+2} T_0^\gamma \int_{|x_0-y| \geq R - \text{diam}(K^1)} \frac{|b(y)|}{|x_0 - y|^{d-1+2\gamma}} dy.$$

By Lemma 2.3 and the dominated convergence theorem, we can choose R large enough such that $I_3 < \frac{\varepsilon}{2}$. Now, we fix the above r, R . Let $R_1 > 0$ so that $K^1 \subset B(0, R_1)$. Since $b \in L_{loc}^1(\mathbb{R}^d)$,

$$\overline{\lim}_{n \rightarrow \infty} I_2 \leq \overline{\lim}_{n \rightarrow \infty} r^{-(d-1)/2} \int_{|y| < R_1 + R} |b(y) - b_n(y)| dy = 0.$$

Then, we have

$$\overline{\lim}_{n \rightarrow \infty} \sup_{(t,x) \in [t_0, T_0] \times K} H_{b-b_n}^\gamma(t, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0.$$

This proves the lemma since ε is arbitrary. \square

Lemma 4.7. Suppose $M > 0$ and $T > 0$. There exist positive constants $C_{20} = C_{20}(d, \alpha, M, T)$ and $C_{21} = C_{21}(d, \alpha, M, T)$ so that for every $n \geq 1, j \geq 1$ and all $a \in (0, M], (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|p_j^{a, b_n}(t, x, y) - p_j^{a, b}(t, x, y)| \leq C_{20} \left(C_{21} M_b(\sqrt{t}) \right)^{j-1} ((H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d, C_{10}/2}^a(t, x, y)). \quad (4.8)$$

Proof. We prove (4.8) inductively in j . Since $\beta \mapsto q_{d, \beta}^a(t, x, y)$ is decreasing, by Theorem 2.1, for all $0 < s < t \leq T$ and $x, z \in \mathbb{R}^d$,

$$p^a(t-s, x, z) \leq C_9 q_{d, C_{10}}^a(t-s, x, z) \leq C_9 q_{d, C_{10}/2}^a(t-s, x, z).$$

We have by (1.7), Theorem 2.2 and (3.1) with $\beta_1 = C_{10}/2$ and $\beta_2 = 3C_{10}/4$,

$$\begin{aligned} &|p_1^{a, b_n}(t, x, y) - p_1^{a, b}(t, x, y)| \\ &= \left| \int_0^t \int_{\mathbb{R}^d} p^a(t-s, x, z) (b(z) - b_n(z)) \nabla_z p^a(s, z, y) dz ds \right| \\ &\leq \int_0^t \int_{\mathbb{R}^d} p^a(t-s, x, z) |b(z) - b_n(z)| |\nabla_z p^a(s, z, y)| dz ds \\ &\lesssim^{C_9 C_{12}} \int_0^t \int_{\mathbb{R}^d} q_{d, C_{10}/2}^a(t-s, x, z) |b(z) - b_n(z)| q_{d+1, 3C_{10}/4}^a(s, z, y) dz ds \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^d} |b(z) - b_n(z)| \left(\int_0^t q_{d,C_{10}/2}^a(t-s, x, z) q_{d+1,3C_{10}/4}^a(s, z, y) ds \right) dz \\
&\stackrel{C_{14}}{\lesssim} \left(\int_{\mathbb{R}^d} |b(z) - b_n(z)| (H^\gamma(t, x, z) + H^\gamma(t, z, y)) dz \right) \cdot q_{d,C_{10}/2}^a(t, x, y) \\
&= (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d,C_{10}/2}^a(t, x, y).
\end{aligned}$$

This proves (4.8) for $j = 1$. Assume (4.8) is true for $j = k \geq 1$. By (1.7),

$$\begin{aligned}
&|p_{k+1}^{a,b_n}(t, x, y) - p_{k+1}^{a,b}(t, x, y)| \\
&\leq \int_0^t \int_{\mathbb{R}^d} |p_k^{a,b_n}(t-s, x, z) - p_k^{a,b}(t-s, x, z)| |b_n(z)| |\nabla_z p^a(s, z, y)| dz ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} |p_k^{a,b}(t-s, x, z)| |b(z) - b_n(z)| |\nabla_z p^a(s, z, y)| dz ds \\
&=: I_1 + I_2.
\end{aligned}$$

Let $C_{20} = C_9 C_{12} C_{14}$ and $C_{21} = 2^{d+3} C_{13} C_{12} C_{14}$. Then

$$\begin{aligned}
I_1 &\stackrel{C_{20}C_{12}}{\lesssim} \int_0^t \int_{\mathbb{R}^d} (C_{21} M_b(\sqrt{t-s}))^{k-1} (H_{b-b_n}^\gamma(t-s, x) + H_{b-b_n}^\gamma(t-s, z)) \\
&\quad \times q_{d,C_{10}/2}^a(t-s, x, z) |b_n(z)| q_{d+1,3C_{10}/4}^a(s, z, y) dz ds \\
&\leq (C_{21} M_b(\sqrt{t}))^{k-1} \int_{\mathbb{R}^d} (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, z)) |b_n(z)| \\
&\quad \times \left(\int_0^t q_{d,C_{10}/2}^a(t-s, x, z) q_{d+1,3C_{10}/4}^a(s, z, y) ds \right) dz \\
&\stackrel{C_{14}}{\lesssim} (C_{21} M_b(\sqrt{t}))^{k-1} \int_{\mathbb{R}^d} (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, z)) |b_n(z)| \\
&\quad \times (H^\gamma(t, x, z) + H^\gamma(t, z, y)) q_{d,C_{10}/2}^a(t, x, y) dz \\
&\leq (C_{21} M_b(\sqrt{t}))^{k-1} \left[H_{b-b_n}^\gamma(t, x) (H_{b_n}^\gamma(t, x) + H_{b_n}^\gamma(t, y)) \right. \\
&\quad \left. + \int_{\mathbb{R}^d} H_{b-b_n}^\gamma(t, z) |b_n(z)| (H^\gamma(t, x, z) + H^\gamma(t, z, y)) dz \right] q_{d,C_{10}/2}^a(t, x, y) \\
&\leq (C_{21} M_b(\sqrt{t}))^{k-1} \left[2C_{13} M_b(\sqrt{t}) H_{b-b_n}^\gamma(t, x) + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(w) - b_n(w)| |b_n(z)| \right. \\
&\quad \left. \times H^\gamma(t, z, w) (H^\gamma(t, x, z) + H^\gamma(t, z, y)) dz dw \right] q_{d,C_{10}/2}^a(t, x, y)
\end{aligned}$$

Note that

$$\begin{aligned}
&H^\gamma(t, z, w) \wedge H^\gamma(t, x, z) \\
&= \left(\frac{1}{|w-z|^{d-1}} \wedge \frac{t^\gamma}{|w-z|^{d-1+2\gamma}} \right) \wedge \left(\frac{1}{|z-x|^{d-1}} \wedge \frac{t^\gamma}{|z-x|^{d-1+2\gamma}} \right) \\
&\stackrel{2^{d+1}}{\lesssim} \left(\frac{1}{|w-x|^{d-1}} \wedge \frac{t^\gamma}{|w-x|^{d-1+2\gamma}} \right) = H^\gamma(t, x, w).
\end{aligned}$$

Similarly,

$$H^\gamma(t, z, w) \wedge H^\gamma(t, y, z) \stackrel{2^{d+1}}{\lesssim} H^\gamma(t, y, w).$$

Thus

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(w) - b_n(w)| |b_n(z)| H^\gamma(t, z, w) (H^\gamma(t, x, z) + H^\gamma(t, z, y)) dz dw$$

$$\begin{aligned}
& \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(w) - b_n(w)| |b_n(z)| \left[H^\gamma(t, x, w) (H^\gamma(t, z, w) + H^\gamma(t, x, z)) \right. \\
& \quad \left. + H^\gamma(t, y, w) (H^\gamma(t, z, w) + H^\gamma(t, y, z)) \right] dz dw \\
& = \int_{\mathbb{R}^d} |b(w) - b_n(w)| \left[H^\gamma(t, x, w) (H_{b_n}^\gamma(t, w) + H_{b_n}^\gamma(t, x)) \right. \\
& \quad \left. + H^\gamma(t, y, w) (H_{b_n}^\gamma(t, w) + H_{b_n}^\gamma(t, y)) \right] dw \\
& \lesssim^{2C_{13}} M_b(\sqrt{t}) \int_{\mathbb{R}^d} |b(w) - b_n(w)| (H^\gamma(t, x, w) + H^\gamma(t, y, w)) dw \\
& = M_b(\sqrt{t}) (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)).
\end{aligned}$$

Therefore

$$\begin{aligned}
I_1 & \lesssim^{C_{20}} \left(C_{21} M_b(\sqrt{t}) \right)^{k-1} 2C_{13} C_{12} C_{14} M_b(\sqrt{t}) \\
& \quad \times \left[H_{b-b_n}^\gamma(t, x) + 2^{d+1} (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) \right] q_{d, C_{10}/2}^a(t, x, y) \\
& \leq \left(2C_{13} C_{12} C_{14} M_b(\sqrt{t}) \right)^k 2^{(d+2)(k-1)} (2^{d+1} + 1) (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d, C_{10}/2}^a(t, x, y).
\end{aligned}$$

On the other hand, by (3.10),

$$\begin{aligned}
I_2 & \lesssim^{C_9 C_{12}} \left(2C_{13} C_{12} C_{14} M_b(\sqrt{t}) \right)^k \int_0^t \int_{\mathbb{R}^d} q_{d, C_{10}/2}^a(t-s, x, z) |b(z) - b_n(z)| q_{d+1, 3C_{10}/4}^a(s, z, y) dz ds \\
& \lesssim^{C_{14}} \int_0^t \int_{\mathbb{R}^d} |b(z) - b_n(z)| (H^\gamma(t, x, z) + H^\gamma(t, x, z)) q_{d, C_{10}/2}^a(t, x, y) dz \\
& = (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d, C_{10}/2}^a(t, x, y).
\end{aligned}$$

Thus

$$\begin{aligned}
& |p_{k+1}^{a, b_n}(t, x, y) - p_{k+1}^{a, b}(t, x, y)| \\
& \lesssim^{C_{20}} \left(2C_{13} C_{12} C_{14} M_b(\sqrt{t}) \right)^k \left(2^{(d+2)(k-1)} (2^{d+1} + 2) \right) (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d, C_{10}/2}^a(t, x, y) \\
& \leq \left(C_{21} M_b(\sqrt{t}) \right)^k (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d, C_{10}/2}^a(t, x, y).
\end{aligned}$$

This completes the proof of the lemma. \square

Lemma 4.8. Suppose $M > 0$ and $T > 0$. For every $a \in (0, M]$, $0 < T_0 < T$ and compact set $K \subset \mathbb{R}^d$, $p^{a, b_n}(t, x, y)$ converges to $p^{a, b}(t, x, y)$ uniformly in $[T_0, T] \times K \times K$ as $n \rightarrow \infty$.

Proof. We divide the proof into two parts. In the first part, we show this lemma holds on $[T_0, T_1] \times K \times K$ for some $T_1 > 0$. In the second part, we prove that for any $k \geq 1$, $p^{a, b_n}(t, x, y)$ converges to $p^{a, b}(t, x, y)$ uniformly in $[kT_1/2, (k+1)T_1/2] \times K \times K$ as $n \rightarrow \infty$. Taking $k = 2\lceil T/T_1 \rceil + 1$ then yields the claim of the lemma.

(i) By (3.18) and Lemma 4.7 with $T = 1$, for all $t \in (0, t_*]$ and $x, y \in K$,

$$\begin{aligned}
& |p^{a, b_n}(t, x, y) - p^{a, b}(t, x, y)| \leq \sum_{k=1}^{\infty} |p_k^{a, b_n}(t, x, y) - p_k^{a, b}(t, x, y)| \\
& \leq C_{20} \sum_{k=1}^{\infty} \left(C_{21} M_b(\sqrt{t}) \right)^k (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d, C_{10}/2}^a(t, x, y).
\end{aligned}$$

Since $b \in \mathbb{K}_{d,1}$, there is a constant $0 < T_1 < t_*$ so that $C_{21}M_b(\sqrt{T_1}) \leq 1/2$. Then for all $t \leq T_1$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} |p^{a,b_n}(t, x, y) - p^{a,b}(t, x, y)| &\leq C_{20} \sum_{k=1}^{\infty} 2^{-(k-1)} (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d,C_{10}/2}^a(t, x, y) \\ &\leq 2C_{20} (H_{b-b_n}^\gamma(t, x) + H_{b-b_n}^\gamma(t, y)) q_{d,C_{10}/2}^a(t, x, y). \end{aligned} \quad (4.9)$$

Without loss of generality, we may and do assume $T_0 < T_1/2$. Note that $q_{d,C_{10}/2}^a(t, x, y) \leq 2T_0^{-d/2}$ for $T_0 \leq t \leq T_1$ and $x, y \in \mathbb{R}^d$. By (4.9) and Lemma 4.6,

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \sup_{t \in [T_0, T_1]} \sup_{x, y \in K} |p^{a,b_n}(t, x, y) - p^{a,b}(t, x, y)| \\ &\leq 4T_0^{-d/2} C_{20} \limsup_{n \rightarrow \infty} \sup_{x, y \in K} (H_{b-b_n}^\gamma(T_1, x) + H_{b-b_n}^\gamma(T_1, y)) = 0. \end{aligned} \quad (4.10)$$

(ii) We prove this part inductively in k . Indeed, it is true when $k = 1$ by Step I. Assume that for any compact set \tilde{K} and $1 \leq k \leq j$, $p^{a,b_n}(t, x, y)$ converges to $p^{a,b}(t, x, y)$ uniformly in $[kT_1/2, (k+1)T_1/2] \times \tilde{K} \times \tilde{K}$ as $n \rightarrow \infty$.

Let $t_1 = T_1/2$. By Chapman-Kolmogorov equation (1.8), For every $t \in [(j+1)T_1, (j+2)T_1/2]$,

$$\begin{aligned} |p^{a,b_n}(t, x, y) - p^{a,b}(t, x, y)| &\leq \int_{\mathbb{R}^d} |p^{a,b_n}(t - t_1, x, z) - p^{a,b}(t - t_1, x, z)| |p^{a,b_n}(t_1, z, y)| dz \\ &\quad + \int_{\mathbb{R}^d} |p^{a,b}(t - t_1, x, z)| |p^{a,b_n}(t_1, z, y) - p^{a,b}(t_1, z, y)| dz \\ &=: I_1 + I_2. \end{aligned}$$

By Theorem 4.1 and (4.7), for every $t \in [(j+1)T_1, (j+2)T_1/2]$,

$$\begin{aligned} |p^{a,b_n}(t - t_1, x, z)| &\leq C_{19} e^{C_{18}(t-t_1)} q_{d,C_{10}^2/(2C_8)}^a(t - t_1, x, z) \leq 2C_{19} e^{C_{18}(j+1)T_1/2} (jT_1/2)^{-d/2}, \\ |p^{a,b}(t - t_1, x, z)| &\leq 2C_{19} e^{C_{18}(j+1)T_1/2} (jT_1/2)^{-d/2}, \end{aligned}$$

and, for any $\varepsilon > 0$, there is a constant $R_0 > 0$ such that for all $n \geq 1$ and $y \in \mathbb{R}^d$,

$$\int_{|z-y| \geq R_0} |p^{a,b_n}(t_1, z, y)| dz < \frac{\varepsilon}{8C_{19} e^{C_{18}(j+1)T_1/2} (jT_1/2)^{-d/2}}.$$

On the other hand, $p^{a,b_n}(t, x, y)$ converges to $p^{a,b}(t, x, y)$ uniformly in $[jT_1/2, (j+1)T_1/2] \times K^{R_0} \times K^{R_0}$ as $n \rightarrow \infty$ since K^{R_0} is bounded. Note that $t - t_1 \in [jT_1/2, (j+1)T_1/2]$. Then for all large enough n ,

$$\sup_{x, z \in K^{R_0}} |p^{a,b_n}(t - t_1, x, z) - p^{a,b}(t - t_1, x, z)| < \frac{\varepsilon}{2C_{17} e^{C_{18}t_1}},$$

while by Theorem 4.1,

$$\int_{\mathbb{R}^d} |p^{a,b_n}(t_1, z, y)| dz \leq C_{17} e^{C_{18}t_1} \int_{\mathbb{R}^d} p^a(2C_8 t_1 / C_{10}, z, y) dz = C_{17} e^{C_{18}t_1}$$

for all $y \in \mathbb{R}^d$. Thus we have for all $(x, y) \in K \times K$

$$\begin{aligned} I_1 &\leq \left(\int_{|z-y| \geq R_0} + \int_{|z-y| < R_0} \right) |p^{a,b_n}(t - t_1, x, z) - p^{a,b}(t - t_1, x, z)| |p^{a,b_n}(t_1, z, y)| dz \\ &\leq 4C_{19} e^{C_{18}(j+1)T_1/2} (jT_1/2)^{-d/2} \cdot \frac{\varepsilon}{8C_{19} e^{C_{18}(j+1)T_1/2} (jT_1/2)^{-d/2}} \end{aligned}$$

$$\begin{aligned}
& + \int_{K^{R_0}} |p^{a,b_n}(t-t_1, x, z) - p^{a,b}(t-t_1, x, z)| |p^{a,b_n}(t_1, z, y)| dz \\
& \leq \frac{\varepsilon}{2} + C_{17} e^{C_{18}t_1} \cdot \frac{\varepsilon}{2C_{17}e^{C_{18}t_1}} \\
& = \varepsilon.
\end{aligned}$$

Similarly, we can get $I_2 < \varepsilon$ for large enough n . Thus, we have proved that $p^{a,b_n}(t, x, y)$ converges to $p^{a,b}(t, x, y)$ uniformly in $[(j+1)T_1, (j+2)T_1/2] \times K \times K$ as $n \rightarrow \infty$. \square

Lemma 4.5 and Lemma 4.8 immediately yield the following.

Lemma 4.9. *Let $M > 0$. For every $a \in (0, M]$,*

$$p^{a,b}(t, x, y) \geq 0, \quad t > 0 \text{ and } x, y \in \mathbb{R}^d.$$

Proof of Theorem 1.2. Theorem 1.2 follows from (3.18), Theorem 3.7, Lemma 4.9, Theorem 3.8 and Theorem 4.3. \square

5 Lower bound estimates

In this section, we derive the sharp lower bound of the heat kernel $p^{a,b}(t, x, y)$. By Lemmas 4.4 and 4.9, $P^{a,b}$ is a Feller semigroup in $C_\infty(\mathbb{R}^d)$. Therefore there is a conservative Feller process $X^{a,b} = \{X_t^{a,b}, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ so that

$$\mathbb{E}_x \left[f(X_t^{a,b}) \right] = P_t^{a,b} f(x) = \int_{\mathbb{R}^d} p^{a,b}(t, x, y) f(y) dy \quad \text{for } x \in \mathbb{R}^d \text{ and } f \in C_\infty(\mathbb{R}^d).$$

The following lemmas will be used to derive the Lévy system of $X^{a,b}$.

Lemma 5.1. *For every $f \in \mathbb{K}_{d,1}$, $\lim_{t \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t P_s^{a,b} |f|(x) ds = 0$.*

Proof. By Theorem 4.1 and (2.2), for $0 < t < 1$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
\int_0^t P_s^{a,b} |f|(x) ds & \leq C_{19} e^{C_{18}t} \int_0^t \int_{\mathbb{R}^d} q_{d, C_{10}^2/(2C_8)}^a(s, x, y) f(y) dy ds \\
& \leq C_{11} C_{19} e^{C_{18}t} \left(\sqrt{t} \int_{\mathbb{R}^d} |f(y)| N_{|f|}^{C_{10}^2/(2C_8)}(t, x, y) dy + \int_0^t \int_{|x-y|^2 \geq s} \frac{a^\alpha s |f|(y)}{|x-y|^{d+\alpha}} dy ds \right) \\
& \leq C_{11} C_{19} e^{C_{18}t} \left(\sqrt{t} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)| N_{|f|}^{C_{10}^2/(2C_8)}(t, x, y) dy + t^{(3-\alpha)/2} H_{|f|}^{(1+\alpha)/2}(t, x) \right).
\end{aligned}$$

Thus, by Lemma 2.4

$$\lim_{t \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t P_s^{a,b} |f|(x) ds \leq \lim_{t \rightarrow 0} C_{11} C_{19} \left(\sqrt{t} N_{|f|}^{C_{10}^2/(2C_8)}(t) + t^{(3-\alpha)/2} \sup_{x \in \mathbb{R}^d} H_{|f|}^{(1+\alpha)/2}(t, x) \right) = 0.$$

\square

Using Lemma 4.4, Lemma 5.1 and Theorem 3.8, the proof of the following result is very similar to that of [6, Theorem 2.5] so it is omitted.

Lemma 5.2. *Suppose $M > 0$. For every $a \in (0, M]$, $x \in \mathbb{R}^d$ and every $f \in C_c^\infty(\mathbb{R}^d)$,*

$$M_t^f := f(X_t^{a,b}) - f(X_0^{a,b}) - \int_0^t \mathcal{L}^{a,b} f(X_s^{a,b}) ds$$

is a martingale under \mathbb{P}_x .

Using above lemma, the proof of next theorem is very similar to that for [7, Lemma 4.7] and [8, Appendix A]; see [6, Theorem 2.6] for some details.

Theorem 5.3. *For $M > 0$ and every $a \in (0, M]$, $X^{a,b}$ has the same Lévy system as Z^a , that is for any $x \in \mathbb{R}^d$, any non-negative measure function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and stopping time S (with respect to the filtration of $X^{a,b}$),*

$$\mathbb{E}_x \left[\sum_{s \leq S} f(s, X_s^{a,b}, X_s^{a,b}) \right] = \mathbb{E}_x \left[\int_0^S \int_{\mathbb{R}^d} f(s, X_s^{a,b}, y) J^a(X_s^{a,b}, y) dy ds \right],$$

For an open set $U \subset \mathbb{R}^d$, define

$$\tau_U^{a,b} := \inf\{t > 0 : X_t^{a,b} \notin U\} \quad \text{and} \quad \sigma_U^{a,b} = \inf\{t \geq 0 : X_t^{a,b} \in U\}.$$

Lemma 5.4. *For each $M > 0$ and $R_0 > 0$, there is a constant $\kappa = \kappa(d, \alpha, M, R_0, b) < 1$ depending on b only via the rate at which $M_b(r)$ goes to zero such that for all $a \in (0, M]$, $r \in (0, R_0]$ and all $x \in \mathbb{R}^d$,*

$$\mathbb{P}_x \left(\tau_{B(x,r)}^{a,b} \leq \kappa r^2 \right) \leq \frac{1}{2}. \quad (5.1)$$

Proof. By the strong Markov property of $X^{a,b}$ (See [3, Exercise (8.17), pp. 43-44]), for $x \in \mathbb{R}^d$ and $t > 0$, we have

$$\begin{aligned} \mathbb{P}_x \left(\tau_{B(x,r)}^{a,b} \leq t \right) &= \mathbb{P}_x \left(\tau_{B(x,r)}^{a,b} \leq t, X_t^{a,b} \in B(x, r/2) \right) + \mathbb{P}_x \left(\tau_{B(x,r)}^{a,b} \leq t, X_t^{a,b} \in B(x, r/2)^c \right) \\ &\leq \mathbb{E}_x \left[\mathbb{P}_{X_{\tau_{B(x,r)}^{a,b}}^{a,b}} \left(\left| X_{t-\tau_{B(x,r)}^{a,b}}^{a,b} - X_0^{a,b} \right| \geq r/2 \right), \tau_{B(x,r)}^{a,b} \leq t \right] \\ &\quad + \mathbb{P}_x \left(\left| X_t^{a,b} - X_0^{a,b} \right| \geq r/2 \right) \\ &\leq 2 \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x \left(\left| X_s^{a,b} - X_0^{a,b} \right| \geq r/2 \right). \end{aligned}$$

By (4.1) with $T = R_0^2$, for $t \in (0, R_0^2]$, there are positive constants $c_i, i = 1, 2, 3$ depending only on d, α, M, R_0 such that

$$\begin{aligned} &\sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x \left(\left| X_s^{a,b} - X_0^{a,b} \right| \geq r/2 \right) \\ &\lesssim c_1 e^{c_2 R_0^2} \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq r/2} \left(s^{-d/2} \exp \left(-\frac{c_3 |x-y|^2}{s} \right) + s^{-d/2} \wedge \frac{a^\alpha s}{|x-y|^{d+\alpha}} \right) dy \\ &\lesssim \omega_d \sup_{s \leq t} \int_{\frac{r}{2\sqrt{s}}}^\infty \left(e^{-c_3 \rho^2} + 1 \wedge \frac{M^\alpha s^{1-\alpha/2}}{\rho^{d+\alpha}} \right) \rho^{d-1} d\rho \\ &\leq \int_{\frac{r}{2\sqrt{t}}}^\infty \left(e^{-c_3 \rho^2} + 1 \wedge \frac{M^\alpha R_0^{2-\alpha}}{\rho^{d+\alpha}} \right) \rho^{d-1} d\rho. \end{aligned}$$

Setting $t = \kappa r^2$ in the last display, where $\kappa \in (0, 1)$ is undetermined, we have

$$\mathbb{P}_x \left(\tau_{B(x,r)}^{a,b} \leq \kappa r^2 \right) \leq 2c_1 e^{c_2 R_0^2} \omega_d \int_{\frac{1}{2\sqrt{\kappa}}}^\infty \left(e^{-c_3 \rho^2} + 1 \wedge \frac{M^\alpha R_0^{2-\alpha}}{\rho^{d+\alpha}} \right) \rho^{d-1} d\rho,$$

which goes to 0 as $\kappa \rightarrow 0$. Thus we can choose $\kappa < 1$ so that (5.1) holds. \square

Lemma 5.5. *For each $M > 0$ and $R_0 > 0$, there is a constant $c_1 = c_1(d, \alpha, M, R_0, b)$ depending on b only via the rate at which $M_b(r)$ goes to zero, such that for all $r \in (0, R_0]$ and $x, y \in \mathbb{R}^d$ with $|x - y| \geq 2r$,*

$$\mathbb{P}_x \left(\sigma_{B(y,r)}^{a,b} < \kappa r^2 \right) \geq c_1 r^{d+2} \frac{a^\alpha}{|x - y|^{d+\alpha}}.$$

Proof. By Lemma 5.4,

$$\mathbb{E}_x \left[\frac{\kappa r^2}{2} \wedge \tau_{B(x,r)}^{a,b} \right] \geq \frac{\kappa r^2}{2} \mathbb{P}_x \left(\tau_{B(x,r)}^{a,b} \geq \frac{\kappa r^2}{2} \right) \geq \frac{\kappa r^2}{4}.$$

By Theorem 5.3, we have

$$\begin{aligned} \mathbb{P}_x \left(\sigma_{B(y,r)}^{a,b} < \kappa r^2 \right) &\geq \mathbb{P}_x \left(X_{\frac{\kappa r^2}{2} \wedge \tau_{B(x,r)}^{a,b}}^{a,b} \in B(y, r) \right) \\ &= \mathbb{E}_x \left(\int_0^{\frac{\kappa r^2}{2} \wedge \tau_{B(x,r)}^{a,b}} \int_{B(y,r)} J^a(X_s^{a,b}, u) du ds \right) \\ &\geq 2^{-(d+\alpha)} \mathbb{E}_x \left[\frac{\kappa r^2}{2} \wedge \tau_{B(x,r)}^{a,b} \right] \int_{B(y,r)} \frac{a^\alpha}{|x - y|^{d+\alpha}} du \\ &\geq \frac{\Omega_d}{4 \cdot 2^{d+\alpha}} \kappa r^{d+2} \frac{a^\alpha}{|x - y|^{d+\alpha}}, \end{aligned}$$

where Ω_d is the volume of unit ball in \mathbb{R}^d , and in the second to the last inequality, we have used the fact that for $u \in B(y, r)$, $|u - X_s^{a,b}| \leq 2r + |x - y| \leq 2|x - y|$. \square

Lemma 5.6. *For every $M > 0$, there is a constant $C_{22} = C_{22}(d, \alpha, M, b)$ depending on b only via the rate at which $M_b(r)$ goes to zero, such that for all $t \in (0, t_*]$, $a \in (0, M]$ and $x, y \in \mathbb{R}^d$*

$$p^{a,b}(t, x, y) \geq C_{22} \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x - y|^{d+\alpha}} \right).$$

Proof. By (3.16), for $t \in (0, t_*]$ and $x, y \in \mathbb{R}^d$, with $|x - y|^2 \leq t$

$$p^{a,b}(t, x, y) \geq C_{16}^{-1} t^{-d/2} \geq C_{16}^{-1} \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x - y|^{d+\alpha}} \right).$$

It remains to consider the case $|x - y|^2 > t$. For any $t \in (0, t_*]$, by the strong Markov property, Lemma 5.4 and Lemma 5.5 with $R_0 = \sqrt{t_*}$ and $r = \sqrt{t}/4$, we have $|x - y| > \sqrt{t} > 2r$ and

$$\begin{aligned} &\mathbb{P}_x \left(X_{\kappa t/16}^{a,b} \in B(y, \sqrt{t}/2) \right) \\ &\geq \mathbb{P}_x \left(X^{a,b} \text{ hits } B(y, \sqrt{t}/2) \text{ before time } \kappa t/16 \text{ and stays there for at least } \kappa t/16 \text{ units of time} \right) \\ &\geq \mathbb{P}_x \left(\sigma_{B(y, \sqrt{t}/4)}^{a,b} \leq \kappa t/16, \tau_{B(y, \sqrt{t}/2)}^{a,b} \circ \theta_{\sigma_{B(y, \sqrt{t}/4)}^{a,b}} \geq \kappa t/16 \right) \\ &\geq \mathbb{P}_x \left(\sigma_{B(y, \sqrt{t}/4)}^{a,b} < \kappa t/16 \right) \inf_{z \in B(y, \sqrt{t}/4)} \mathbb{P}_z \left(\tau_{B(y, \sqrt{t}/2)}^{a,b} \geq \kappa t/16 \right) \\ &\geq \mathbb{P}_x \left(\sigma_{B(y, \sqrt{t}/4)}^{a,b} < \kappa t/16 \right) \inf_{z \in B(y, \sqrt{t}/4)} \mathbb{P}_z \left(\tau_{B(z, \sqrt{t}/4)}^{a,b} \geq \kappa t/16 \right) \\ &\geq c_1 t^{(d+2)/2} \frac{a^\alpha}{|x - y|^{d+\alpha}}, \end{aligned}$$

for some constant $c_1 = c_1(d, \alpha, M, b) > 0$. Combining this with Lemma 5.4 and Chapman-Kolmogorov equation (1.8), we have for $t \in (0, t_*]$

$$\begin{aligned}
p^{a,b}(t, x, y) &= \int_{\mathbb{R}^d} p^{a,b}(\kappa t/16, x, z) p^{a,b}((1 - \kappa/16)t, z, y) dz \\
&\geq \int_{B(y, \sqrt{t}/2)} p^{a,b}(\kappa t/16, x, z) p^{a,b}((1 - \kappa/16)t, z, y) dz \\
&\geq \inf_{z \in B(y, \sqrt{t}/2)} p^{a,b}((1 - \kappa/16)t, z, y) \mathbb{P}_x \left(X_{\kappa t/16}^{a,b} \in B(y, \sqrt{t}/2) \right) \\
&\geq c_2 t^{-d/2} t^{(d+2)/2} \frac{a^\alpha}{|x - y|^{d+\alpha}} \\
&= c_2 \frac{a^\alpha t}{|x - y|^{d+\alpha}} \geq c_2 \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x - y|^{d+\alpha}} \right),
\end{aligned}$$

where $c_2 = c_2(d, \alpha, M, b)$ is a positive constant and in the third to the last inequality, we have used the fact that $\kappa < 1$ and for $z \in B(y, \sqrt{t}/2)$, $|z - y|^2 < t/4 < (1 - \kappa/16)t$. \square

Lemma 5.7. *Suppose $M > 0$. For all $a \in (0, M]$, $t \in (0, t_*]$ and $x, y \in \mathbb{R}^d$, there are constants $C_i = C_i(d, \alpha, M) > 0, i = 23, 24$ such that*

$$p^{a,b}(t, x, y) \geq C_{23} t^{-d/2} \exp \left(-\frac{C_{24} |x - y|^2}{t} \right).$$

Proof. By (3.16), for all $t \in (0, t_*]$ and $x, y \in \mathbb{R}^d$ with $|x - y|^2 < t$, we have

$$p^{a,b}(t, x, y) \geq C_{16}^{-1} t^{-d/2}. \quad (5.2)$$

Next, we consider the case $|x - y|^2 > t$. We fix $x, y \in \mathbb{R}^d$ with $|x - y|^2 \geq t$. Let k be the smallest integer such that $9|x - y|^2/t < k$. Set $\xi_j = x + \frac{j-1}{k}(y - x)$, $1 \leq j \leq k-1$ and $A = \prod_{j=1}^{k-1} B(\xi_j, \frac{\sqrt{t}}{3\sqrt{k}})$.

For any $(x_1, \dots, x_{k-1}) \in A$, we have $|x - x_1| < \frac{\sqrt{t}}{3\sqrt{k}} < \frac{\sqrt{t}}{\sqrt{k}}$,

$$\max_{1 \leq j \leq k-1} |x_j - x_{j-1}| = \max_{1 \leq j \leq k-1} \left| x_j - \xi_j + \xi_{j-1} - x_{j-1} + \frac{y - x}{k} \right| < \frac{\sqrt{t}}{3\sqrt{k}} + \frac{\sqrt{t}}{3\sqrt{k}} + \frac{\sqrt{t}}{3\sqrt{k}} = \frac{\sqrt{t}}{\sqrt{k}}$$

and $|x_{k-1} - y| = |x_{k-1} - \xi_{k-1} + \xi_{k-1} - y| < \frac{\sqrt{t}}{\sqrt{k}}$. Hence by Lemma 4.9, Chapman-Kolmogorov equation (1.8) and (5.2),

$$\begin{aligned}
p^{a,b}(t, x, y) &= \int_{\mathbb{R}^{d(k-1)}} p^{a,b}\left(\frac{t}{k}, x, x_1\right) \cdots p^{a,b}\left(\frac{t}{k}, x_{k-1}, y\right) dx_1 dx_2 \cdots dx_{k-1} \\
&\geq \int_A p^{a,b}\left(\frac{t}{k}, x, x_1\right) \cdots p^{a,b}\left(\frac{t}{k}, x_{k-1}, y\right) dx_1 dx_2 \cdots dx_{k-1} \\
&\geq C_{16}^{-k} \left(\frac{t}{k}\right)^{-dk/2} \Omega_d^{k-1} \left(\frac{\sqrt{t}}{3\sqrt{k}}\right)^{d(k-1)} \\
&= t^{-d/2} \frac{k^{d/2}}{C_{16}} \left(\frac{k^{d/2}}{C_{16}} \frac{\Omega_d}{3^d k^{d/2}}\right)^{k-1} \\
&= \frac{k^{d/2}}{C_{16}} t^{-d/2} \left(\frac{\Omega_d}{C_{16} 3^d}\right)^{k-1} \\
&\geq \frac{3^d}{C_{16}} t^{-d/2} \exp \left(-\ln \frac{C_{16} 3^d}{\Omega_d} \frac{9|x - y|^2}{t} \right),
\end{aligned}$$

where Ω_d is the volume of unit ball in \mathbb{R}^d . This together with (5.2) proves the lemma with $C_{23} := \frac{3^d}{C_{16}}$ and $C_{24} := 9 \ln \frac{C_{16} 3^d}{\Omega_d}$. \square

Proof of Theorem 1.3. The upper bound of $p^{a,b}(t, x, y)$ is shown by Theorem 4.1 and Lemma 4.9. We need only to show the lower bound. Without loss of generality, we assume $T > t_*$. If $t \in (0, t_*]$, by Lemma 5.6 and Lemma 5.7, there is a constant $c_1 = c_1(d, \alpha, M, b) > 0$ such that for $x, y \in \mathbb{R}^d$

$$\begin{aligned} p^{a,b}(t, x, y) &\geq \frac{1}{2} \left(C_{23} t^{-d/2} \exp \left(-\frac{C_{24}|x-y|^2}{t} \right) + C_{22} \left(t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}} \right) \right) \\ &\geq c_1 \left(t^{-d/2} \exp \left(-\frac{C_{24}|x-y|^2}{t} \right) + t^{-d/2} \wedge \frac{a^\alpha t}{|x-y|^{d+\alpha}} \right) \\ &= c_1 q_{d,C_{24}}^a(t, x, y). \end{aligned} \quad (5.3)$$

If $t \geq t_*$, we let k be the smallest integer such that $t_* k \geq t > (k-1)t_*$. Note that by Theorem 2.1, for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$q_{d,C_{24}}^a(t, x, y) \geq \left(\frac{C_{10}}{C_{24}} \right)^{d/2} q_{d,C_{10}}^a \left(\frac{C_{10}t}{C_{24}}, x, y \right) \geq \left(\frac{C_{10}}{C_{24}} \right)^{d/2} C_9^{-1} p^a \left(\frac{C_{10}t}{C_{24}}, x, y \right).$$

Using this, (1.8), (5.3) and Theorem 2.1, we have

$$\begin{aligned} p^{a,b}(t, x, y) &\geq c_1^{-k} \int_{\mathbb{R}^{d(k-1)}} q_{d,C_{24}}^a \left(\frac{t}{k}, x, x_1 \right) \cdots q_{d,C_{24}}^a \left(\frac{t}{k}, x_{k-1}, y \right) dx_1 \cdots dx_{k-1} \\ &\geq c_1^{-k} \left(\frac{C_{10}}{C_{24}} \right)^{dk/2} C_9^{-k} \int_{\mathbb{R}^{d(k-1)}} p^a \left(\frac{C_{10}t}{C_{24}k}, x, x_1 \right) \cdots p^a \left(\frac{C_{10}t}{C_{24}k}, x_{k-1}, y \right) dx_1 \cdots dx_{k-1} \\ &= c_1^{-k} \left(\frac{C_{10}}{C_{24}} \right)^{dk/2} C_9^{-k} p^a \left(\frac{C_{10}t}{C_{24}}, x, y \right) \\ &\geq \frac{C_{10}C_7}{c_1 C_7} \left(\frac{C_{10}}{c_1 C_7} \right)^{d(k-1)/2} q_{d,C_8}^a \left(\frac{C_{10}t}{C_{24}}, x, y \right) \\ &\geq \frac{C_{10}C_7c_2}{c_1 C_7} \left(\frac{C_{10}}{c_1 C_7} \right)^{dT/(2t_*)} q_{d,C_8C_{24}/C_{10}}^a(t, x, y) \\ &\geq \frac{C_{10}C_7c_2}{c_1 C_7} \left(\frac{C_{10}}{c_1 C_7} \right)^{dT/(2t_*)} q_{d,C_8C_{24}/C_{10}}^a(t, x, y). \end{aligned}$$

where $c_2 = c_2(d, \alpha, M, b)$ is a positive constant. This completes the proof. \square

6 Martingale problem and Lévy process with drift

Following the approach in [11], we can show that the martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ is well-posed, and there is a unique weak solution to SDE (1.1).

For $a > 0$ and $\lambda > 0$, define

$$u_\lambda^a(x) = \int_0^\infty e^{-\lambda t} p^a(t, x) dt, \quad x \in \mathbb{R}^d.$$

Lemma 6.1. *There is a constants $C_{25} = C_{25}(d)$ such that for all $a > 0$, $\lambda \geq 1$ and $x \in \mathbb{R}^d$, we have*

$$u_\lambda^a(x) \leq C_{25}(1 \vee a^\alpha) \begin{cases} \frac{1}{|x|^{d-1}} \wedge \frac{\lambda^{-\frac{\alpha+1}{2}}}{|x|^{d+\alpha}}, & d = 2, \\ \frac{1}{|x|^{d-2}} \wedge \frac{\lambda^{-\frac{\alpha+2}{2}}}{|x|^{d+\alpha}}, & d > 2, \end{cases} \quad (6.1)$$

and

$$|\nabla u_\lambda^a(x)| \leq C_{25}(1 \vee a^\alpha) \left(\frac{1}{|x|^{d-1}} \wedge \frac{\lambda^{-\frac{\alpha+2}{2}}}{|x|^{d+1+\alpha}} \right). \quad (6.2)$$

Proof. Note that for each $\theta > 0$, the function $\psi(t) = t^\theta e^{-t}$ on $[0, \infty)$ is bounded by $\theta^\theta e^{-\theta}$. By (1.5), we have

$$\begin{aligned}
u_\lambda^a(x) &\leq C_1 \int_0^\infty e^{-\lambda t} \left(t^{-d/2} e^{-C_2|x|^2/t} + (a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|x|^{d+\alpha}} \right) dt \\
&\leq c_1 \int_0^\infty e^{-\lambda t} \left(\frac{t}{|x|^{d+2}} + \frac{a^\alpha t}{|x|^{d+\alpha}} \right) dt \\
&= c_1 \lambda^{-2} \left(\frac{1}{|x|^{d+2}} + \frac{a^\alpha}{|x|^{d+\alpha}} \right) \\
&\leq c_1 (1 \vee a^\alpha) \lambda^{-2} \left(\frac{1}{|x|^{d+2}} + \frac{1}{|x|^{d+\alpha}} \right).
\end{aligned} \tag{6.3}$$

Since $\lambda \geq 1$, if $|x|^2 \geq 1/\lambda$,

$$u_\lambda^a(x) \leq 2c_1 (1 \vee a^\alpha) \frac{\lambda^{-\frac{\alpha+2}{2}}}{|x|^{d+\alpha}}. \tag{6.4}$$

When $|x|^2 < 1/\lambda$, similar to (6.3), we have

$$\int_0^{|x|^2} e^{-\lambda t} p^a(t, x) dt \leq c_1 \int_0^{|x|^2} \left(\frac{t}{|x|^{d+2}} + \frac{a^\alpha t}{|x|^{d+\alpha}} \right) dt \leq \frac{c_1}{2} \left(\frac{1}{|x|^{d-2}} + \frac{a^\alpha}{|x|^{d-4+\alpha}} \right) \leq \frac{c_1 (1 \vee a^\alpha)}{|x|^{d-2}} \tag{6.5}$$

and

$$\begin{aligned}
\int_{|x|^2}^\infty e^{-\lambda t} p^a(t, x) dt &\leq C_1 \int_{|x|^2}^\infty e^{-\lambda t} t^{-d/2} dt \\
&\leq C_1 \begin{cases} \frac{1}{|x|} \int_{|x|^2}^\infty e^{-t} t^{-1/2} dt \leq \frac{\sqrt{\pi}}{|x|} & \text{if } d = 2, \\ \int_{|x|^2}^\infty t^{-d/2} dt = \frac{2}{d-2} \frac{1}{|x|^{d-2}} & \text{if } d > 2. \end{cases}
\end{aligned} \tag{6.6}$$

Therefore, (6.1) follows from (6.4)-(6.6). Finally, (6.2) follows from (2.5) and (6.1). \square

For $a > 0$ and $\lambda > 0$, define the resolvent operator U_λ^a by

$$U_\lambda^a g(x) = \int_{\mathbb{R}^d} u_\lambda^a(x-y) g(y) dy = \int_{\mathbb{R}^d} u_\lambda^a(y) g(x-y) dy, \quad g \in C_b(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Let $C_\infty^\infty(\mathbb{R}^d)$ be the collection of the smooth functions on \mathbb{R}^d that together with their partial derivatives of any order vanish at infinity.

Lemma 6.2. *For every $a > 0$ and $\lambda \geq 1$, U_λ^a and ∇U_λ^a are bounded operators on $C_\infty^\infty(\mathbb{R}^d)$. Moreover, $U_\lambda^a f \in C_\infty^\infty(\mathbb{R}^d)$ for every $f \in C_\infty^\infty(\mathbb{R}^d)$.*

Proof. By (6.2), we have for every $a > 0$, $\lambda \geq 1$, $f \in C_\infty^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\nabla u_\lambda^a(y)| |f(x-y)| dy \leq C_{25} (1 \vee a^\alpha) \|f\|_\infty \int_{\mathbb{R}^d} \frac{1}{|y|^{d-1}} \wedge \frac{\lambda^{-\frac{\alpha+2}{2}}}{|y|^{d+1+\alpha}} dy < \infty.$$

Combining this with the fact that u_λ^a is continuously differentiable off the origin and the dominated convergence theorem, we have

$$\nabla U_\lambda^a f(x) = \int_{\mathbb{R}^d} \nabla u_\lambda^a(x-y) f(y) dy = \int_{\mathbb{R}^d} \nabla u_\lambda^a(y) f(x-y) dy.$$

Since both u_λ^a and ∇u_λ^a are integrable over \mathbb{R}^d and $f(x-y)$ converges to 0 as $|x| \rightarrow \infty$, we have that both $U_\lambda^a f$ and $\nabla U_\lambda^a f$ are in $C_\infty(\mathbb{R}^d)$ and

$$\|U_\lambda^a f\|_\infty \leq C_{25}(1 \vee a^\alpha)\|f\|_\infty, \text{ and } \|\nabla U_\lambda^a f\|_\infty \leq C_{25}(1 \vee a^\alpha)\|f\|_\infty,$$

where C_{25} is the constant from Lemma 6.1. Similarly, by the dominated convergence theorem, for $f \in C_\infty^\infty(\mathbb{R}^d)$, we have

$$\partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} U_\lambda^a f(x) = \int_{\mathbb{R}^d} u_\lambda^a(y) \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} f(x-y) dy,$$

which shows that $U_\lambda^a f \in C_\infty^\infty(\mathbb{R}^d)$. \square

Lemma 6.3. *Suppose that $M > 0$ and $b \in \mathbb{K}_{d,1}$. There is a constant $\lambda_0 = \lambda_0(d, \alpha, M, b) \geq 1$ with the dependence on b only via the rate at which $M_b(r)$ goes to zero such that for every $a \in (0, M]$, $\lambda \geq \lambda_0$ and $f \in C_\infty(\mathbb{R}^d)$,*

$$\|\nabla U_\lambda^a(bf)\|_\infty \leq \frac{1}{2}\|f\|_\infty.$$

Proof. By (2.7)(with $\beta = \frac{\alpha+2}{2}$) and (6.2), we have for $a \in (0, M]$, $\lambda \geq \lambda_0$ and $f \in C_\infty(\mathbb{R}^d)$,

$$\begin{aligned} \|\nabla U_\lambda^a(bf)\|_\infty &\leq C_{25}(1 \vee a^\alpha) \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1}{|x-y|^{d-1}} \wedge \frac{\lambda^{-\frac{\alpha+2}{2}}}{|x-y|^{d+1+\alpha}} \right) |b(y)| |f(y)| dy \\ &\leq C_{25} c_1 \|f\|_\infty (1 \vee M^\alpha) M_b(\lambda^{-1/2}). \end{aligned}$$

Since $b \in \mathbb{K}_{d,1}$, we can choose $\lambda_0 \geq 1$ such that $C_{25} c_1 (1 \vee M^\alpha) M_b(\lambda^{-1/2}) \leq 1/2$ for every $\lambda > \lambda_0$. This completes the proof. \square

By (4.1) for $\lambda > C_{18} C_{10}/(2C_8)$,

$$\begin{aligned} \mathbb{E}_x \left[\int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] &\leq C_{17} C_{10}/(2C_8) \int_{\mathbb{R}^d} \int_0^\infty e^{-(\lambda - C_{18} C_{10}/(2C_8))t} p^a(t, x, y) dt |b(y)| dy \\ &= C_{17} C_{10}/(2C_8) \int_{\mathbb{R}^d} u_{\lambda - C_{18} C_{10}/(2C_8)}^a(x-y) |b(y)| dy. \end{aligned}$$

Similar to Lemma 6.3, by (6.1), there is a constant $C_{26} > C_{18} C_{10}/(2C_8) \vee 1$ so that for every $a \in (0, M]$ and $\lambda > C_{26}$,

$$\sup_{x \in \mathbb{R}^d} U_\lambda^a |b|(x) = \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] < \infty. \quad (6.7)$$

By increasing the value of λ_0 in Lemma 6.3 if needed, we may and do assume that $\lambda_0 \geq C_{26}$.

Theorem 6.4 (Uniqueness). *For each $x \in \mathbb{R}^d$ and $a \in (0, M]$, \mathbb{P}_x is the unique solution to the martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ with initial value x .*

Proof. Recall that X_t be the coordinate map on $\mathbb{D}([0, \infty), \mathbb{R}^d)$. Using Lemmas 6.1-6.3 and (6.7), we can finish the proof by repeating the arguments in the proof of [11, Theorem 2.3] except using the following Itô's formula in place of that in Step (ii) of [11, Theorem 2.3]:

$$\begin{aligned} e^{-\lambda t} f(X_t) &= f(X_0) + \int_0^t e^{-\lambda s} dM_s^f + \int_0^t e^{-\lambda s} \left(\Delta f(X_s) + a^\alpha \Delta^{\alpha/2} f(X_s) + b(X_s) \cdot \nabla f(X_s) \right) ds \\ &\quad - \lambda \int_0^t e^{-\lambda s} f(X_s) ds. \end{aligned}$$

\square

Proof of Theorem 1.4. Theorem 6.4 implies that the martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ is well-posed. The rest follows from Theorem 1.2. \square

The following theorem establishes the existence of the weak solution of SDE (1.1).

Theorem 6.5 (Existence). *For every $a > 0$, there is a process Z^a defined on Ω so that all its paths are right continuous and admit left limits, and*

$$X_t^{a,b} = x + Z_t^a + \int_0^t b(X_s^{a,b}) ds, \quad t \geq 0.$$

Proof. The proof is almost the same to that of [11, Theorem 3.1], except that we use the following arguments instead of those at the beginning of Page 13 in [11]: for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} & \int_0^t b(X_s^{a,b}) \nabla f(X_s^{a,b}) ds + \int_0^t \Delta f(X_s^{a,b}) ds \\ &= \int_0^t \nabla f(X_s^{a,b}) dA_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s^{a,b}) d\langle M^i, M^j \rangle_s, \end{aligned}$$

which implies that

$$A_t = \int_t^\infty b(X_s^{a,b}) ds \quad \text{and} \quad \langle M^i, M^j \rangle_t = \delta_{ij} t.$$

Here $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. \square

Proof of Theorem 1.5. The existence of weak solution to SDE (1.1) follows from Lemma 6.5. Every weak solution to (1.1) solves the martingale problem for $(\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d))$ by Itô's formula. Then, the rest follows from Theorem 1.4. \square

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